

The Baum-Connes conjecture for a quantum semidirect product

RUBÉN MARTOS

ABSTRACT. The well known “associativity property” of the crossed product by a semidirect product of discrete groups is generalized into the context of discrete *quantum* groups. This decomposition allows us to define an appropriate triangulated functor in such a way that if $\mathbb{F} = \Gamma \ltimes_{\alpha} \mathbb{G}$ is a quantum semidirect product where Γ is a discrete group and \mathbb{G} is a compact quantum group, then we prove that the (quantum) Baum-Connes property for $\hat{\mathbb{F}}$ is equivalent to the Baum-Connes property for Γ generalizing (and simplifying) thus the result [4] of J. Chabert. Finally, we observe that the same method can be applied for a *compact bicrossed product* in the sense of [8]. In both cases, the K -amenability connexion between $\hat{\mathbb{F}}$ and Γ is investigated.

1. INTRODUCTION

The Baum-Connes conjecture has been formulated in 1982 by P. Baum and A. Connes. We still do not know any counter example to the original conjecture but it is known that the one with coefficients is false. For this reason we refer to the Baum-Connes conjecture with coefficients as the *Baum-Connes property*. The goal of the conjecture was to understand the link between two operator K -groups of different nature that would establish a strong connexion between geometry and topology in a more abstract and general index-theory context. More precisely, if G is a (second countable) locally compact group and A is a (separable) G - C^* -algebra, then the Baum-Connes property for G with coefficients in A claims that the assembly map

$$\mu_A^G : K_*^{top}(G; A) \longrightarrow K_*(G \ltimes_r A)$$

is an isomorphism, where $K_*^{top}(G; A)$ is the equivariant K -homology with compact support of G with coefficients in A and $K_*(G \ltimes_r A)$ is the K -theory of the reduced crossed product $G \ltimes_r A$. This property has been proved for a large class of groups; let us mention the remarkable work of N. Higson and G. Kasparov [9] about groups with Haagerup property and the one of V. Lafforgue [11] about hyperbolic groups.

The equivariant K -homology with compact support $K_*^{top}(G; A)$ is, of course, the geometrical object obtained from the classifying space of proper actions of G , thus it is, *a priori*, the easiest group to calculate than the group $K_*(G \ltimes_r A)$ which is the

one of analytical nature and then less flexible in its structure. Nevertheless, sometimes the group $K_*^{top}(G; A)$ creates non-trivial troubles. That's why R. Meyer and R. Nest provide in 2006 a new formulation of the Baum-Connes property in a well-suited category framework [14]. More precisely, if now \mathcal{KK}^G is the G -equivariant Kasparov category and $F(A) := K_*(G \rtimes_r A)$ is the homological functor over \mathcal{KK}^G defining the right-hand side of the Baum-Connes assembly map, then Meyer and Nest show in [14] that the assembly map μ_A^G is equivalent to the natural transformation

$$\eta_A^G : \mathbb{L}F(A) \longrightarrow F(A),$$

where $\mathbb{L}F$ is the localisation of the functor F with respect to an appropriated complementary pair of (localizing) subcategories $(\mathcal{L}, \mathcal{N})$; namely, \mathcal{L} is the subcategory of \mathcal{KK}^G of *compactly induced G - C^* -algebras* and \mathcal{N} is the subcategory of \mathcal{KK}^G of *compactly contractible G - C^* -algebras*.

This reformulation allows, particularly, to avoid any geometrical construction and thus to replace G by a locally compact *quantum* group \mathbb{G} . The problem is the torsion structure of such a \mathbb{G} . Indeed, if G is a discrete group, its torsion is completely described in terms of the finite subgroups of G whereas for the case in which $\hat{\mathbb{G}}$ is a discrete quantum group, the notion of torsion is a non trivial problem that has been introduced firstly by R. Meyer and R. Nest (see [15] and [13]) and recently re-interpreted by Y. Arano and K. De Commer in terms of fusion rings (see [1]). But torsion phenomena in the quantum setting are far from been completely understood, so that the current proper formulation of the *quantum* Baum-Connes property concerns only *torsion free discrete quantum groups* $\hat{\mathbb{G}}$.

In this article, we study the stability of the Baum-Connes property under the semidirect product construction. Namely, consider a semidirect product of locally compact groups $F := \Gamma \rtimes G$ such that F is equipped with a γ -element and for every compact subgroup $\Lambda < \Gamma$ the semidirect product $F_\Lambda = \Lambda \rtimes G$ satisfies the Baum-Connes property. In this situation, J. Chabert shows in [4] that if the Baum-Connes property with coefficients holds for Γ , so it does for F . The strategy consists in using the canonical $*$ -isomorphism $F \rtimes_r A \cong \Gamma \rtimes_r (G \rtimes_r A)$ for any F - C^* -algebra A with the goal of constructing, in a natural way, partial descent homomorphisms and thus to translate the assembly map for F into an assembly map for Γ through the transitional group G . The assumption of the existence of a γ -element for F is actually unnecessary (as it's shown in [5]). In fact, all technical problems we have to deal with, in order to get such a translation, appear in the treatment of the equivariant K -homology with compact support in relation with the associativity above. So that, our strategy is to apply the Meyer-Nest machinery thanks to which we can avoid all these shortcomings and the conclusion becomes much more clear and proper: the Baum-Connes property with coefficients holds for F if and only if it holds for Γ (Theorem 5.5). Actually, the assumption about the semidirect products F_Λ is an unnecessary one since it is only an auxiliary assumption by which we get an identification that, in our categorical framework, is automatically fulfilled (see Corollary 5.4).

Our goal is to generalize this result when we have a *quantum* semidirect product $\mathbb{F} := \Gamma \ltimes \mathbb{G}$ where Γ is a discrete group acting by quantum automorphisms in the compact quantum group \mathbb{G} (such a construction is due to S. Wang, see [20], that is a special case of a compact bicrossed product of a matched pair of a discrete group and a compact quantum group). Observe that the $*$ -isomorphism $F \ltimes_r A \cong \Gamma \ltimes_r (G \ltimes_r A)$ in the classical case is the key tool to reach all subsequent constructions; therefore our main technical problem is to obtain such associativity in the quantum setting. Namely, we prove there exists a canonical $*$ -isomorphism $\hat{\mathbb{F}} \ltimes_r A \cong \Gamma \ltimes_r (\hat{\mathbb{G}} \ltimes_r A)$ for any $\hat{\mathbb{F}}$ - C^* -algebra A (Theorem 5.1). With this aim in mind, it's convenient to analyse the structure of the reduced crossed products by a discrete (quantum) group and, in this sense, we establish an universal property for such a crossed product (Theorem 2.3.1) which will be very useful throughout all the article.

To conclude, we observe that the same method can be applied to a *compact bicrossed product* in the sense of [8]. Namely, if $\mathbb{F} = \Gamma \bowtie G$ is such a compact bicrossed product associated to a matched pair of groups (Γ, G) with Γ discrete and G compact, then the main difference with respect to the quantum semidirect product case is the existence of the action of Γ on $\hat{G} \ltimes_r A$. Once this action is established, all the subsequent constructions and arguments can be “copied” and so the analogous result always holds (see Theorem 6.6).

Notice however that, according to the above discussion about the torsion phenomena of quantum groups, we need $\hat{\mathbb{F}}$ to be torsion-free in order to give a meaning to the *quantum* Baum-Connes property. In this way, we give a more precise picture of the torsion nature of such a quantum semidirect product and obtain the natural stabilisation property we expect: if Γ and $\hat{\mathbb{G}}$ are torsion-free, so it is $\hat{\mathbb{F}}$ (Theorem 4.2). We observe as well that the same arguments can be applied to a *compact bicrossed product* in the sense of [8] (Theorem 6.4). In any case, we can firstly do the simple observation that, with the structure of the irreducible representation theory of such a $\mathbb{F} = \Gamma \ltimes \mathbb{G}$ or $\mathbb{F} = \Gamma \bowtie G$, we can easily establish a natural decomposition of the fusion ring of \mathbb{F} that allows thus to get the analogous stabilisation property for *strong* torsion-freeness (see Proposition 4.1 and Proposition 6.3, respectively).

A last property of own interest is studied: K -amenability. Namely, whenever \mathbb{F} is a quantum semidirect product or a compact bicrossed product, we prove that $\hat{\mathbb{F}}$ is K -amenable if and only if Γ is K -amenable (see Theorem 5.6 and Theorem 6.7 and notice that for the quantum semidirect product we need in addition the co-amenableity of \mathbb{G} as assumption).

ACKNOWLEDGEMENTS. I would like to thank sincerely my advisor Pierre Fima because of his very useful remarks, comments and rectifications.

2. PRELIMINARY RESULTS

2.1. Notations

First of all, let us fix the notations we use throughout the whole article.

We denote by $\mathcal{B}(H)$ the space of all linear operators of the Hilbert space H and by $\mathcal{L}_A(H)$ the space of all adjointable operators of the Hilbert A -module H . All our C^* -algebras are supposed to be *separables*. We use systematically the leg and Sweedler notations. We denote by $\mathcal{A}b$ the abelian category of abelian groups. The symbol \otimes stands for the minimal tensor product of C^* -algebras. If M and N are two R -modules for some ring R , the symbol \odot stands for their algebraic tensor product over R , and we write $M \odot_R N$.

If H is a finite dimensional Hilbert space, we denote by \overline{H} its dual or conjugate vector space, so that if $\{\xi_1, \dots, \xi_{\dim(H)}\}$ is an orthonormal basis for H and $\{\omega_1, \dots, \omega_{\dim(H)}\}$ its dual basis in \overline{H} , we denote by $*$ the usual homomorphism between H and \overline{H} such that $\xi_i^* = \omega_i$, for all $i = 1, \dots, \dim(H)$.

If $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a compact quantum group, the set of all unitary equivalence classes of irreducible unitary finite dimensional representations of \mathbb{G} is denoted by $Irr(\mathbb{G})$. The trivial representation of \mathbb{G} is denoted by ϵ . If $x \in Irr(\mathbb{G})$ is such a class, we write $w^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$ for a representative of x and H_x for the finite dimensional Hilbert space on which w^x acts (we write $\dim(x) \equiv n_x$ for the dimension of H_x). The linear span of matrix coefficients of all finite dimensional representations of \mathbb{G} is denoted by $Pol(\mathbb{G})$. Given $x, y \in Irr(\mathbb{G})$, the tensor product of x and y is denoted by $x \oplus y$. Given $x \in Irr(\mathbb{G})$, there exists a unique class of irreducible unitary finite dimensional representation of \mathbb{G} denoted by \overline{x} such that $Mor(\epsilon, x \oplus \overline{x}) \neq 0 \neq Mor(\epsilon, \overline{x} \oplus x)$ and it is called contragredient or conjugate representation of x . In this way, there exists an antilinear isomorphism $J_x : H_x \longrightarrow H_{\overline{x}}$. We define the operator $Q_x := J_x^* J_x$, which is an invertible positive self-adjoint operator unique up to multiplication of a real number. We choose this number such that J_x is normalized meaning that $Tr(J_x^* J_x) = Tr((J_x^* J_x)^{-1})$. Thus, the quantum dimension of a class $x \in Irr(\mathbb{G})$ is defined by $\dim_q(x) = Tr(Q_x)$. The fundamental unitary of \mathbb{G} is denoted by $W_{\mathbb{G}}$. The Haar state of \mathbb{G} is denoted by $h_{\mathbb{G}}$ and the corresponding GNS construction by $(L^2(\mathbb{G}), \lambda, \Omega)$. For more details of these definitions and constructions we send the reader to [21].

If $\alpha : A \longrightarrow M(c_0(\hat{\mathbb{G}}) \otimes A)$ is an action of $\hat{\mathbb{G}}$ on a C^* -algebra A and $x \in Irr(\mathbb{G})$, we write $\alpha^x(a) := \alpha(a)(p_x \otimes id_A) \in \mathcal{B}(H_x) \otimes A$, for all $a \in A$ where p_x is the central projection of $c_0(\hat{\mathbb{G}})$ over $\mathcal{B}(H_x)$ so that $\alpha(a) = \bigoplus_{x \in Irr(\mathbb{G})}^{\oplus_{c_0}} \alpha^x(a)$. If $\{\xi_1^x, \dots, \xi_{n_x}^x\}$ is an orthonormal basis of H_x and ω_{ξ_i, ξ_j} is the linear form of $\mathcal{B}(H_x)$ defined by $\omega_{\xi_i, \xi_j}(T) := \langle T(\xi_j), \xi_i \rangle$ for all $T \in \mathcal{B}(H_x)$, we define $\alpha_{i,j}^x(a) := (\omega_{\xi_i^x, \xi_j^x} \otimes id_A)(\alpha^x(a)) \in A$, for all $a \in A$ and all $i, j = 1, \dots, n_x$. So that, if $\{m_{i,j}^x\}_{i,j=1, \dots, n_x}$ are the matrix units in $\mathcal{B}(H_x)$ associated to the basis $\{\xi_1^x, \dots, \xi_{n_x}^x\}$, then we have $\alpha^x(a) = \sum_{i,j=1}^{n_x} m_{i,j}^x \otimes \alpha_{i,j}^x(a)$. In an analogous way, if $U \in M(c_0(\hat{\mathbb{G}}) \otimes C)$ for some C^* -algebra C , then we define $U^x := U(p_x \otimes id_C) \in \mathcal{B}(H_x) \otimes C$

and $U_{i,j}^x := (\omega_{\xi_i^x, \xi_j^x} \otimes id_C)U^x \in C$, for all $i, j = 1 \dots, n_x$.

We use systematically the well known one-to-one correspondence between unitary representations $U \in M(c_0(\hat{\mathbb{G}}) \otimes C)$ and non-degenerate $*$ -homomorphisms $\phi_U : C_m(\mathbb{G}) \longrightarrow M(C)$.

Let (A, δ) be a right \mathbb{G} - C^* -algebra; the fixed points space of A will be denoted by A^δ and we say that δ is a *torsion action* if δ is ergodic and A is unital finite dimensional. Given an irreducible representation $x \in Irr(\mathbb{G})$, we put $K_x := \{X \in \overline{H}_x \otimes A \mid (id \otimes \delta)(X) = [X]_{12}[w^x]_{13}\}$ and use systematically the natural identification $K_x \cong Mor(x, \delta)$, with $Mor(x, \delta) := \{T : H_x \longrightarrow A \mid T \text{ is linear such that } \delta(T(\xi)) = (T \otimes id_{C(\mathbb{G})})w^x(\xi \otimes 1_{C(\mathbb{G})})\}$. The corresponding x -spectral subspace of A is denoted by \mathcal{A}_x so that the corresponding Podles subalgebra of A is denoted by $\mathcal{A}_{\mathbb{G}}$. By abuse of language, both K_x and $Mor(x, \delta)$ are called *spectral subspaces* as well. Finally, from the general theory of spectral subspaces (see [7] for the details) we recall that each x -spectral subspace \mathcal{A}_x is finite dimensional with $dim(\mathcal{A}_x) \leq dim_q(x)$ whenever δ is ergodic; and that there always exists a conditional expectation $\mathbb{E}_\delta : A \longrightarrow A^\delta$ which is faithful over $\mathcal{A}_{\mathbb{G}}$ so that we have the following decomposition $\mathcal{A}_{\mathbb{G}} = \bigoplus_{x \in Irr(\mathbb{G})} \mathcal{A}_x$.

For our purposes it's convenient to introduce the following operations in $\mathcal{A}_{\mathbb{G}}$: given irreducible representations of \mathbb{G} , say $x, y, z \in Irr(\mathbb{G})$, we define the following element $X \otimes_\Phi Y := ([X]_{13}[Y]_{23})(\Phi \otimes 1_A) \in \overline{H}_z \otimes A$, for all $X \in K_x$ and $Y \in K_y$ and all intertwiner $\Phi \in Mor(z, x \oplus y)$; where $[X]_{13}$ and $[Y]_{23}$ are the corresponding legs of X and Y in $\overline{H}_x \otimes \overline{H}_y \otimes A$. It's straightforward to check that $X \otimes_\Phi Y \in K_z$.

Given now an irreducible representation $x \in Irr(\mathbb{G})$, fix an orthonormal basis $\{\xi_1^x, \dots, \xi_{dim(x)}^x\}$ of H_x that diagonalises the canonical operator Q_x and let $\{\omega_1^x, \dots, \omega_{dim(x)}^x\}$ be its dual basis in the dual space \overline{H}_x . In this case, the intertwiner Φ_x can be written in coordinates under the expression $\Phi_x = \sum_{i=1}^{n_x} \sqrt{\lambda_i^x} \xi_i^x \otimes \omega_i^x$, where $\lambda_i^x \in \mathbb{R}^+$ is the eigenvalue of Q_x associated to the vector ξ_i^x , for each $i = 1, \dots, n_x \equiv dim(x)$. In this situation, if $X \in K_x$ written in coordinates under the form $X = \sum_{i=1}^{n_x} \omega_i^x \otimes a_i$ as element of $\overline{H}_x \otimes A$, for some $a_i \in A$, for all $i = 1, \dots, n_x$, we define the following element $X^\# := \sum_{i=1}^{n_x} J_x(\xi_i^x)^* \otimes a_i^* \in \overline{H}_x \otimes A$. A straightforward computation shows that the association $X \longmapsto X^\#$ is antilinear and that $X^\# \in K_{\overline{x}}$.

2.1.1 Remark. We can do thus the multiplication over Φ_x and over $\Phi_{\overline{x}}$ defined above. Namely, the previous constructions show that $X \otimes_{\Phi_x} X^\# \in K_\epsilon$ and $X^\# \otimes_{\Phi_{\overline{x}}} X \in K_\epsilon$. Observe that $K_\epsilon = A^\delta$. Hence, if δ is an ergodic action, we'll have that $X \otimes_{\Phi_x} X^\#$ and $X^\# \otimes_{\Phi_{\overline{x}}} X$ will be scalar multiples of 1_A . It will be useful to write down explicit formulas in coordinates for these products: $X \otimes_{\Phi_x} X^\# = \sum_{i=1}^{n_x} \lambda_i^x a_i a_i^*$; $X^\# \otimes_{\Phi_{\overline{x}}} X = \sum_{i=1}^{n_x} \lambda_i^x a_i^* a_i$.

Concerning these constructions, the following technical observation will help to conclude later on our torsion analysis for a quantum semidirect product.

2.1.2 Lemma. *Let \mathbb{G} be a compact quantum group and (A, δ) a right ergodic \mathbb{G} - C^* -algebra. Given irreducible representations $x, y \in \text{Irr}(\mathbb{G})$ and non-zero elements $X \in K_x$, $Y \in K_y$; then there exist an irreducible representation $z \in \text{Irr}(\mathbb{G})$ and an intertwiner $\Phi \in \text{Mor}(z, x \oplus y)$ such that $X \otimes_{\Phi} Y \neq 0$.*

Proof. Let's fix orthonormal basis $\{\xi_1^x, \dots, \xi_{\dim(x)}^x\}$ of H_x and $\{\xi_1^y, \dots, \xi_{\dim(y)}^y\}$ of H_y that diagonalise the canonical operators $Q_x = J_x^* J_x$ and $Q_y = J_y^* J_y$, respectively; with eigenvalues $\{\lambda_i^x\}_{i=1, \dots, n_x}$ and $\{\mu_j^y\}_{j=1, \dots, n_y}$, respectively. Denote by $\{\omega_1^x, \dots, \omega_{\dim(x)}^x\}$ and $\{\omega_1^y, \dots, \omega_{\dim(y)}^y\}$ the corresponding dual basis of \overline{H}_x and \overline{H}_y , respectively.

Suppose that for all irreducible representation $z \in \text{Irr}(\mathbb{G})$ and all intertwiner $\Phi \in \text{Mor}(z, x \oplus y)$ we have $X \otimes_{\Phi} Y = 0$, that is, $X \otimes_{\Phi} Y = \sum_{i,j} \omega_i^x \otimes \omega_j^y \circ \Phi \otimes a_i b_j = 0$ where we use the coordinate expressions for X and Y as above.

Multiplying by $\Phi^* \otimes 1_A$, this is still zero,

$$\sum_{i,j} \omega_i^x \otimes \omega_j^y \circ \Phi \Phi^* \otimes a_i b_j = 0 \quad (2.1)$$

This is true for *every* irreducible representation $z \in \text{Irr}(\mathbb{G})$ and *every* intertwiner $\Phi \in \text{Mor}(z, x \oplus y)$. Let's consider the decomposition in direct sum of irreducible representations of $x \oplus y$, say $\{z_l\}_l$; and denote by $\{p_l\}_l \subset \mathcal{B}(H_x \otimes H_y)$ the corresponding family of mutually orthogonal finite-dimensional projections with sum $\text{id}_{H_x \otimes H_y}$. Likewise, for every $k = 1, \dots, \dim(\text{Mor}(z, x \oplus y))$ consider the corresponding intertwiners $\Phi_k \in \text{Mor}(z_l, x \oplus y)$ for each l which are such that $\Phi_k^* \Phi_k = \text{id}$ and $\sum_{k=1}^{\dim(\text{Mor}(z, x \oplus y))} \Phi_k \Phi_k^* = p_k$.

Hence we consider the identity (2.1) above for these intertwiners Φ_k for each $k = 1, \dots, \dim(\text{Mor}(z, x \oplus y))$ and next we sum over k . We get $\sum_{i,j} \omega_i^x \otimes \omega_j^y p_k \otimes a_i b_j = 0$. Next, we can sum over l and we get $\sum_{i,j} \omega_i^x \otimes \omega_j^y \otimes a_i b_j = 0$, what implies that $a_i b_j = 0$, for all $i = 1, \dots, n_x$ and all $j = 1, \dots, n_y$.

On the other hand, by assumption our action δ is ergodic so that

$$\sum_{i=1}^{n_x} \lambda_i^x a_i^* a_i = X^{\#} \otimes_{\Phi_x} X = \lambda 1_A \text{ and } Y \otimes_{\Phi_y} Y^{\#} = \sum_{j=1}^{n_y} \mu_j^y b_j b_j^* = \mu 1_A, \text{ for some } \lambda, \mu \in \mathbb{C} \setminus \{0\}.$$

Using the consequence of our last identity, we get $0 \neq \lambda \mu = 0$; a contradiction. \blacksquare

2.2. Fusion rings and *strong* torsion-freeness

The notion of torsion-freeness for a discrete quantum group was initially introduced by R. Meyer and R. Nest (see [15] and [13]) as follows. We say that a discrete quantum group $\hat{\mathbb{G}}$ is torsion-free if any torsion action of \mathbb{G} is \mathbb{G} -equivariantly Morita equivalent to the trivial \mathbb{G} - C^* -algebra \mathbb{C} .

2.2.1 Remark. If $\hat{\mathbb{G}}$ is a discrete quantum group that has a non-trivial finite discrete quantum subgroup, then $\hat{\mathbb{G}}$ is *not* torsion-free because the co-multiplication of such a non-trivial finite discrete quantum group Λ would define an ergodic action of \mathbb{G} on $C(\Lambda)$.

Recently [1], Y. Arano and K. De Commer have re-interpreted this notion in terms of fusion rings giving a *strong* torsion-freeness definition that implies the Meyer-Nest ones. In order to write down this last definition, let us recall briefly the corresponding fusion rings theory (we refer to [1] for all the details and further properties).

Let $(I, \mathbb{1})$ be an involutive pointed set and J any set. Let $(\mathbb{Z}_I, \oplus, \otimes)$ be a fusion ring with fusion rules given by the collection of natural numbers $\{N_{\beta, \gamma}^\alpha\}_{\alpha, \beta, \gamma \in I}$ and $(\mathbb{Z}_J, \oplus, \otimes)$ a J -based co-finite module with respect to \mathbb{Z}_I defined by a collection of natural numbers $\{N_{\alpha, b}^c\}_{\alpha \in I, b, c \in J}$. We say that $(\mathbb{Z}_I, \oplus, \otimes)$ is trivial or zero if $I = \{\mathbb{1}\}$.

We say that $(\mathbb{Z}_J, \oplus, \otimes)$ is *connected* if for any $b, c \in J$, there exists $\alpha \in I$ such that $N_{\alpha, b}^c \neq 0$. Remark that $(\mathbb{Z}_I, \oplus, \otimes)$ is a natural I -based (connected) co-finite module with respect to itself with left \otimes -multiplication and bilinear form given by $\langle \alpha, \beta \rangle := \alpha \otimes \beta$, for all $\alpha, \beta \in I$. It's called *the standard I -based module*.

If $\hat{\mathbb{G}}$ be a discrete quantum group, we denote by $Fus(\hat{\mathbb{G}})$ the usual fusion ring of $\hat{\mathbb{G}}$ given by the irreducible representations of \mathbb{G} . If $(\mathbb{Z}_{I_1}, \oplus, \otimes, d_1)$ and $(\mathbb{Z}_{I_2}, \oplus, \otimes, d_2)$ are two fusion rings, we define the tensor product of \mathbb{Z}_{I_1} and \mathbb{Z}_{I_2} as the free \mathbb{Z} -module $\mathbb{Z}_{I_1} \odot_{\mathbb{Z}} \mathbb{Z}_{I_2}$ which is naturally a fusion ring denoted by $\mathbb{Z}_{I_1} \otimes \mathbb{Z}_{I_2}$.

In this situation, a fusion ring $(\mathbb{Z}_I, \oplus, \otimes, d)$ is called torsion-free if any based connected co-finite module is isomorphic to the standard based module. In particular, if \mathbb{G} is a compact quantum group, we say that $\hat{\mathbb{G}}$ is strong torsion-free if $Fus(\hat{\mathbb{G}})$ is torsion-free.

Observe that torsion-freeness of fusion rings is *not* preserved in general by tensor product and we have (see [1] for a proof):

2.2.2 Theorem. *Let $(\mathbb{Z}_{I_1}, \otimes)$ and $(\mathbb{Z}_{I_2}, \otimes)$ be torsion-free fusion rings and assume that $(\mathbb{Z}_{I_1} \odot \mathbb{Z}_{I_2}, \otimes)$ is not torsion-free. Then $(\mathbb{Z}_{I_1}, \otimes)$ and $(\mathbb{Z}_{I_2}, \otimes)$ have non-trivial isomorphic finite fusion subrings.*

2.3. Quantum crossed products

We recall here the crossed product construction. The next result is well known to specialist. Since we could not find any reference in the litterature, we include the complete proof for the readers convenience.

2.3.1 Theorem-Definition. *Let \mathbb{G} be a compact quantum group and (A, α) a $\hat{\mathbb{G}}$ - C^* -algebra.*

There exists a unique (up to a canonical isomorphism) C^ -algebra P with a non-degenerate $*$ -homomorphism $\pi : A \longrightarrow P$, a unitary representation $U \in M(c_0(\hat{\mathbb{G}}) \otimes P)$ and a non-degenerate completely positive KSGNS-faithful map $E : P \longrightarrow M(A)$ such that*

- i) $\pi(a)U_{i,j}^x = \sum_{k=1}^{\dim(x)} U_{i,k}^x \pi(\alpha_{k,j}^x(a)),$ for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \dots, \dim(x)$.
- ii) $P = C^* \langle \pi(a)U_{i,j}^x : a \in A, x \in Irr(\mathbb{G}), i, j = 1, \dots, \dim(x) \rangle$
- iii) $E(\pi(a)U_{i,j}^x) = \delta_{x, \epsilon} a$ for all $x \in Irr(\mathbb{G})$ and all $a \in A$.

iv) For any C^* -algebra Q with a triple (ρ, V, E') where $\rho : A \longrightarrow Q$ is a non-degenerate $*$ -homomorphism, $V \in M(c_0(\hat{\mathbb{G}}) \otimes Q)$ is a unitary representation and $E' : Q \longrightarrow M(A)$ is a strict completely positive KSGNS-faithful map satisfying the analogous properties (i), (ii) and (iii) above, then there exists a (necessarily unique) $*$ -isomorphism $\psi : P \longrightarrow Q$ such that

$$\psi(\pi(a)U_{i,j}^x) = \rho(a)V_{i,j}^x,$$

for all $x \in \text{Irr}(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \dots, \dim(x)$. Moreover, E' is a non-degenerate map and we have $E = E' \circ \psi$.

The C^* -algebra P constructed in this way is called reduced crossed product of A by $\hat{\mathbb{G}}$ and is denoted by $\hat{\mathbb{G}} \ltimes_{\alpha,r} A$.

Proof. P will be a sub- C^* -algebra of $\mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$.

For the non-degenerate $*$ -homomorphism π we consider the representation of A on $L^2(\mathbb{G}) \otimes A$ “twisting” by the action α . Precisely, take the GNS representation $(L^2(\mathbb{G}), \hat{\lambda}, \Omega)$ associated to the left Haar weight \hat{h}_L of $\hat{\mathbb{G}}$. So we have that $\hat{\lambda} \otimes id_A : c_0(\hat{\mathbb{G}}) \otimes A \longrightarrow \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$ is a non-degenerate $*$ -homomorphism. We define the non-degenerate $*$ -homomorphism $\pi : A \longrightarrow \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$ by $\pi(a) := (\hat{\lambda} \otimes id_A) \circ \alpha(a)$, for all $a \in A$.

For U we consider the unitary representation of $\hat{\mathbb{G}}$ on $L^2(\mathbb{G}) \otimes A$ induced by λ . Precisely, take the fundamental unitary $W = \bigoplus_{x \in \text{Irr}(\mathbb{G})}^{c_0} w^x \in M(c_0(\hat{\mathbb{G}}) \otimes C_r(\mathbb{G}))$ with

$w^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$, for all $x \in \text{Irr}(\mathbb{G})$. We define the unitary $U := (id_{c_0(\hat{\mathbb{G}})} \otimes \lambda)(W) \otimes id_A \in M(c_0(\hat{\mathbb{G}}) \otimes \mathcal{L}_A(L^2(\mathbb{G}) \otimes A))$.

A straightforward computation yields the following expressions

$$U^x = (id_{\mathcal{B}(H_x)} \otimes \lambda)w^x \otimes id_A \in \mathcal{B}(H_x) \otimes \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$$

$$U_{i,j}^x = \lambda(w_{i,j}^x) \otimes id_A \in \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$$

for all $x \in \text{Irr}(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$.

In this situation, we can check the formula $\pi(a)U_{i,j}^x = \sum_{k=1}^{n_x} U_{i,k}^x \pi(\alpha_{k,j}^x(a))$, for all $x \in$

$Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \dots, \dim(x)$. Indeed,

$$\begin{aligned}
\sum_{k=1}^{n_x} U_{i,k}^x \pi(\alpha_{k,j}^x(a)) &= \sum_{k=1}^{n_x} (\omega_{e_i^x, e_k^x} \otimes id) \left((id \otimes \lambda)(W) \otimes id_A(p_x \otimes id) \right) (\hat{\lambda} \otimes id_A) \circ \alpha(\alpha_{k,j}^x(a)) \\
&= \sum_{k=1}^{n_x} (\omega_{e_i^x, e_k^x} \otimes id) \left((\hat{\lambda} \otimes \lambda)(W) \otimes id_A(p_x \otimes id) \right) (\alpha(\alpha_{k,j}^x(a))) \\
&= \sum_{k=1}^{n_x} (\omega_{e_i^x, e_k^x} \otimes id) (p_x \otimes id) (\hat{\lambda} \otimes \lambda)(W) \otimes id_A(\alpha(\alpha_{k,j}^x(a))) \\
&= \sum_{k=1}^{n_x} (\omega_{e_i^x, e_k^x} \otimes id) (\omega_{e_k^x, e_j^x} \otimes id) ((p_x \otimes id) (\hat{\lambda} \otimes \lambda)(W) \otimes id_A(id \otimes \alpha)\alpha(a)(p_x \otimes id_A)) \\
&\stackrel{*}{=} \sum_{k=1}^{n_x} (\omega_{e_i^x, e_k^x} \otimes id) (\omega_{e_k^x, e_j^x} \otimes id) ((p_x \otimes id) (\hat{\lambda} \otimes \lambda)(W) \otimes id_A(\hat{\Delta} \otimes id_A)\alpha(a)(p_x \otimes id_A)) \\
&\stackrel{**}{=} \sum_{k=1}^{n_x} (\omega_{e_i^x, e_k^x} \otimes id) (\omega_{e_k^x, e_j^x} \otimes id) ((1 \otimes \alpha(a)) (\hat{\lambda} \otimes \lambda)(W) \otimes id_A(p_x \otimes id)) \\
&= (\omega_{e_i^x, e_j^x} \otimes id \otimes id) ((1 \otimes \alpha(a)) (\hat{\lambda} \otimes \lambda)(W) \otimes id_A(p_x \otimes id)) \\
&= (\hat{\lambda} \otimes id_A)\alpha(a)(\omega_{e_i^x, e_j^x} \otimes id) ((id \otimes \lambda)(W) \otimes id_A(p_x \otimes id)) = \pi(a)U_{i,j}^x
\end{aligned}$$

where the equality (*) holds because α is an action of $\hat{\mathbb{G}}$ over A and the equality (**) holds because of the definition of the co-multiplication $\hat{\Delta}$ of $\hat{\mathbb{G}}$ in terms of its fundamental unitary.

Thus we define $P := C^* \langle \pi(a)U_{i,j}^x : a \in A, x \in Irr(\mathbb{G}), i, j = 1, \dots, \dim(x) \rangle$ which is a sub- C^* -algebra of $\mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$.

To conclude the construction of P as in the statement, we have to define a non-degenerate completely positive KSGNS-faithful map $E : P \longrightarrow M(A) = \mathcal{L}_A(A)$ satisfying the formula $E(\pi(a)U_{i,j}^x) = a\delta_{x,\epsilon}$ for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \dots, \dim(x)$. Namely, let us define the linear map $\Upsilon : A \longrightarrow L^2(\mathbb{G}) \otimes A$ by $\Upsilon(a) := \Omega \otimes a$, for all $a \in A$. It is actually an adjointable map between A and $L^2(\mathbb{G}) \otimes A$ whose adjoint is such that $\Upsilon^*(\lambda(w_{i,j}^x)\Omega \otimes a) = h_{\mathbb{G}}(w_{i,j}^x)a$, for all $x \in Irr(\mathbb{G})$, all $i, j = 1, \dots, \dim(x)$ and all $a \in A$. Thus $E(X) := \Upsilon^* \circ X \circ \Upsilon$, for all $X \in P$ defines a completely positive map from P into $M(A)$.

We assure that the triple $(L^2(\mathbb{G}) \otimes A, id, \Upsilon)$ is the KSGNS construction for E . We only have to prove that $\overline{L^2(\mathbb{G}) \otimes A} = \overline{span\{P\Upsilon(A)\}}$; but, by construction, it suffices to show that $\lambda(w_{i,j}^x)\Omega \otimes a \in \overline{P\Upsilon(A)}$ for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$; which is straightforward.

Finally, an easy computation shows that the formula $E(\pi(a)U_{i,j}^x) = a\delta_{x,\epsilon}$ holds for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \dots, \dim(x)$. For, fix an orthonormal basis $\{\xi_1^x, \dots, \xi_{n_x}^x\}$ of H_x diagonalizing the canonical operator Q_x with eigenvalues $\{\lambda_j^x\}_{j=1, \dots, n_x}$, so that the formula $\lambda(w_{i,j}^x)\Omega = \frac{\sqrt{\lambda_j^x}}{\sqrt{\dim_q(x)}} \xi_i^x \otimes \omega_j^x$ holds for all $i, j = 1, \dots, n_x$ where

$\{\omega_1^x, \dots, \omega_{n_x}^x\}$ is the dual basis of $\{\xi_1^x, \dots, \xi_{n_x}^x\}$ in the dual space $H_{\overline{x}}$. We write

$$\begin{aligned}
E(\pi(a)U_{i,j}^x)(b) &= \Upsilon^*(\pi(a)U_{i,j}^x(\Upsilon(b))) = \Upsilon^*(\pi(a)U_{i,j}^x(\Omega \otimes b)) \\
&= \Upsilon^*(\pi(a)(\lambda(w_{i,j}^x) \otimes id_A)(\Omega \otimes b)) \\
&= \Upsilon^*(\pi(a)(\lambda(w_{i,j}^x)\Omega \otimes b)) \\
&= \Upsilon^*((\hat{\lambda} \otimes id_A) \circ \alpha(a)(\lambda(w_{i,j}^x)\Omega \otimes b)) \\
&= \Upsilon^*((p_x \otimes id_A) \left[(\hat{\lambda} \otimes id_A) \circ \alpha(a) \right] (p_x \otimes id_A)(\lambda(w_{i,j}^x)\Omega \otimes b)) \\
&= \Upsilon^*((p_x \otimes id_A) \left[(\hat{\lambda} \otimes id_A) \circ \alpha^x(a) \right] (\lambda(w_{i,j}^x)\Omega \otimes b)) \\
&= \Upsilon^*((p_x \otimes id_A) \left[(\hat{\lambda} \otimes id_A) \circ \sum_{i,j=1}^{n_x} m_{i,j}^x \otimes \alpha_{i,j}^x(a) \right] \\
&\quad \left(\left(\frac{\sqrt{\lambda_j^x}}{\sqrt{\dim_q(x)}} \xi_i^x \otimes \omega_j^x \right) \otimes b \right)) \\
&= \Upsilon^*\left(\sum_{i,j=1}^{n_x} (m_{i,j}^x \otimes id_{H_{\overline{x}}} \otimes \alpha_{i,j}^x(a)) \left(\left(\frac{\sqrt{\lambda_j^x}}{\sqrt{\dim_q(x)}} \xi_i^x \otimes \omega_j^x \right) \otimes b \right) \right) \\
&= \Upsilon^*\left(\frac{\sqrt{\lambda_j^x}}{\sqrt{\dim_q(x)}} \sum_{i,j=1}^{n_x} \delta_{j,i} \xi_i^x \otimes \omega_j^x \otimes \alpha_{i,j}^x(a) b \right) \\
&= \Upsilon^*\left(\frac{\sqrt{\lambda_i^x}}{\sqrt{\dim_q(x)}} \xi_i^x \otimes \omega_i^x \otimes \alpha_{i,i}^x(a) b \right) = \Upsilon^*(\lambda(w_{i,i}^x)\Omega \otimes \alpha_{i,i}^x(a) b) \\
&= h_{\mathbb{G}}(w_{i,i}^x) \alpha_{i,i}^x(a) b = \alpha_{i,i}^x(a) b \delta_{x,\epsilon} = ab \delta_{x,\epsilon},
\end{aligned}$$

where we've used the orthogonality relations (and the definition of the KSGNS construction). Since it is true for all $b \in B$, we conclude the required formula.

Observe that by KSGNS construction, E is just a *strict* completely positive map (see [12] for the details). But, thanks to the property $E(\pi(a)) = a$, for all $a \in A$ we've just proved, it's clear that E is actually a *non-degenerate* completely positive map as assured in the statement.

And now, let us establish the uniqueness of such a construction. Suppose Q is another C^* -algebra with a triple (ρ, V, E') where $\rho : A \rightarrow Q$ is a non degenerate $*$ -homomorphism, $V \in M(c_0(\hat{\mathbb{G}}) \otimes Q)$ is a unitary representation and $E' : Q \rightarrow M(A)$ is a strict completely positive KSGNS-faithful map satisfying the analogous properties (i), (ii) and (iii) of the triple (π, U, E) associated to P . We have to show that there exists a (unique) $*$ -isomorphism $\psi : P \rightarrow Q$ such that $\psi(\pi(a)U_{i,j}^x) = \rho(a)V_{i,j}^x$, for all $x \in Irr(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \dots, \dim(x)$.

Given the strict completely positive KSGNS-faithful maps $E : P \rightarrow M(A)$ and $E' : Q \rightarrow M(A)$, consider their KSGNS constructions; say $(L^2(\mathbb{G}) \otimes A, id, \Upsilon)$ et (K, σ, Υ') , respectively. This means in particular that $L^2(\mathbb{G}) \otimes A = \overline{\text{span}\{P\Upsilon(A)\}}$, $\sigma : Q \rightarrow \mathcal{L}_A(K)$

is a non-degenerate faithful $*$ -homomorphism such that $K = \overline{\text{span}\{\sigma(Q)\Upsilon'(A)\}}$ and that $E'(Y) = (\Upsilon')^* \circ \sigma(Y) \circ \Upsilon'$, for all $Y \in Q$.

Define a unitary operator $\mathcal{U} : L^2(\mathbb{G}) \otimes A \longrightarrow K$. If such an operator exists, it must verify the formula $\mathcal{U}(X\Upsilon(b)) = \sigma(Y)\Upsilon'(b)$, for all $X = \pi(a)U_{i,j}^x \in P$, $Y = \rho(a)V_{i,j}^x \in Q$ and all $b \in A$.

Actually, a straightforward computation shows that the formula above defines an isometry. Indeed, doing the identification $Q \cong \sigma(Q)$ (by virtue of the faithfulness of the KSGNS construction), let's take $X = \pi(a)U_{i,j}^x$, $X' = \pi(a')U_{i,j}^{x'} \in P$, $Y = \rho(a)V_{i,j}^x$, $Y' = \rho(a')V_{i,j}^{x'} \in Q$, $b, b' \in A$ and write

$$\begin{aligned}
& \langle \mathcal{U}(X\Upsilon(b)), \mathcal{U}(X'\Upsilon(b')) \rangle = \langle Y\Upsilon'(b), Y'\Upsilon'(b') \rangle \\
& = \langle \rho(a)V_{i,j}^x \Upsilon'(b), \rho(a')V_{i,j}^{x'} \Upsilon'(b') \rangle = \langle \Upsilon'(b), (V_{i,j}^x)^* \rho(a^*) \rho(a') V_{i,j}^{x'} \Upsilon'(b') \rangle \\
& = \langle b, (\Upsilon')^* ((V_{i,j}^x)^* \rho(a^* a') V_{i,j}^{x'} \Upsilon'(b')) \rangle = \langle b, E'((V_{i,j}^x)^* \rho(a^* a') V_{i,j}^{x'})(b') \rangle \\
& = \langle b, E' \left(\sum_{k=1}^{n_{x'}} \rho(\alpha_{k,j}^{x'}(a^* a')) (V_{i,k}^x)^* V_{i,j}^{x'} \right) (b') \rangle \stackrel{*}{=} \langle b, E' \left(\sum_t \rho(\alpha_{t,s}^{x' \oplus \epsilon}(a^* a')) V_{r,t}^{\bar{x} \oplus x'} \right) (b') \rangle \\
& = \langle b, \sum_t \alpha_{t,s}^{x' \oplus \epsilon}(a^* a') \delta_{\bar{x} \oplus x', \epsilon} b' \rangle = \langle b, E \left(\sum_t \pi(\alpha_{t,s}^{x' \oplus \epsilon}(a^* a')) U_{r,t}^{\bar{x} \oplus x'} \right) (b') \rangle \\
& = \langle b, E \left(\sum_{k=1}^{n_{x'}} \pi(\alpha_{k,j}^{x'}(a^* a')) (U_{i,k}^x)^* U_{i,j}^{x'} \right) (b') \rangle = \langle b, E((U_{i,j}^x)^* \pi(a^* a') U_{i,j}^{x'})(b') \rangle \\
& = \langle b, \Upsilon^*((U_{i,j}^x)^* \pi(a^* a') U_{i,j}^{x'} \Upsilon(b')) \rangle = \langle \Upsilon(b), (U_{i,j}^x)^* \pi(a^* a') U_{i,j}^{x'} \Upsilon(b') \rangle \\
& = \langle \Upsilon(b), (U_{i,j}^x)^* \pi(a^*) \pi(a') U_{i,j}^{x'} \Upsilon(b') \rangle = \langle \pi(a) U_{i,j}^x \Upsilon(b), \pi(a') U_{i,j}^{x'} \Upsilon(b') \rangle \\
& = \langle X\Upsilon(b), X'\Upsilon(b') \rangle,
\end{aligned}$$

where it should be noticed that in $(*)$ we use the index notation $r \equiv (i, i)$, $t \equiv (k, j)$ and $s \equiv (j, j)$ in order to write down properly the coefficients for the tensor product of irreducible representations under the form above.

Doing again the identification $Q \cong \sigma(Q)$, we define

$$\begin{aligned}
\psi : P & \longrightarrow Q \\
X & \longmapsto \psi(X) := \mathcal{U} \circ X \circ \mathcal{U}^*
\end{aligned}$$

It's clear that ψ is a $*$ -isomorphism and the formula $\psi(\pi(a)U_{i,j}^x) = \rho(a)V_{i,j}^x$ for all $x \in \text{Irr}(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \dots, \dim(x)$ is easily checked.

Moreover, by assumption we have $E'(\rho(a)V_{i,j}^x) = a\delta_{\gamma,e}$ for all $x \in \text{Irr}(\mathbb{G})$, all $a \in A$ and all $i, j = 1, \dots, \dim(x)$ and so $E'(\rho(a)) = a$ for all $a \in A$; then it's clear that E' is in fact a *non-degenerate* map. On the other hand, the relation $E = E' \circ \psi$ holds by construction. \blacksquare

Applying the universal property of the preceding theorem, we get

2.3.2 Corollary. *Let \mathbb{G} be a compact quantum group. If (A, α) is a $\hat{\mathbb{G}}$ - C^* -algebra and B is any C^* -algebra, then we have a canonical $*$ -isomorphism*

$$\hat{\mathbb{G}} \rtimes_{id \otimes \alpha, r} (B \otimes A) \cong B \otimes \hat{\mathbb{G}} \rtimes_{\alpha, r} A$$

2.1 Remark. Let \mathbb{G} be a compact quantum group, (A, α) , (B, β) two $\hat{\mathbb{G}}$ - C^* -algebras and $\varphi : A \longrightarrow B$ a $\hat{\mathbb{G}}$ -equivariant $*$ -homomorphism. In this situation, there exists a $*$ -homomorphism $\mathcal{Z}(\varphi) \equiv id \rtimes \varphi : \hat{\mathbb{G}} \rtimes_{\alpha, r} A \longrightarrow \hat{\mathbb{G}} \rtimes_{\beta, r} B$ such that $\mathcal{Z}(\varphi)(\pi_\alpha(a)(U^\alpha)_{i,j}^x) = \pi_\beta(\varphi(a))(U^\beta)_{i,j}^x$, for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$ where $(\pi_\alpha, U^\alpha, E_\alpha)$ and $(\pi_\beta, U^\beta, E_\beta)$ are the canonical triples associated to the reduced crossed products $\hat{\mathbb{G}} \rtimes_{\alpha, r} A$ and $\hat{\mathbb{G}} \rtimes_{\beta, r} B$, respectively (recall Theorem 2.3.1).

The $*$ -homomorphism $\mathcal{Z}(\varphi)$ above is nothing but the restriction of $\mathcal{L}_A(L^2(\mathbb{G}) \otimes A) \longrightarrow \mathcal{L}_B(L^2(\mathbb{G}) \otimes B)$ defined by $T \longmapsto \mathcal{U}_\varphi(T \otimes_\varphi id_B) \mathcal{U}_\varphi^{-1}$, for all $T \in \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$, where $\mathcal{U}_\varphi : L^2(\mathbb{G}) \otimes A \otimes_\varphi B \xrightarrow{\sim} L^2(\mathbb{G}) \otimes B$ is the canonical isometry of Hilbert modules such that $\mathcal{U}_\varphi(\xi \otimes a \otimes_\varphi b) = \xi \otimes \varphi(a)b$, for all $\xi \in L^2(\mathbb{G})$, all $a \in A$ and all $b \in B$.

Finally, observe that $\mathcal{Z}(\varphi) = id \rtimes \varphi$ is, by construction, compatible with the elements of the canonical triples in the following sense

$$\mathcal{Z}(\varphi)(\pi_\alpha(a)) = \pi_\beta(\varphi(a)), \mathcal{Z}(\varphi)((U^\alpha)_{i,j}^x) = (U^\beta)_{i,j}^x, E_\beta \circ \mathcal{Z}(\varphi) = E_\alpha \circ \varphi,$$

for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$.

2.2 Proposition. *Let \mathbb{G} be a compact quantum group and (A, α) , (B, β) two $\hat{\mathbb{G}}$ - C^* -algebras. If $\varphi : A \longrightarrow B$ is any $\hat{\mathbb{G}}$ -equivariant $*$ -homomorphism, then there exists a canonical $*$ -isomorphism*

$$\hat{\mathbb{G}} \rtimes_r C_\varphi \cong C_{id \rtimes \varphi},$$

where C_φ denotes the cone of the $*$ -homomorphism φ and $C_{id \rtimes \varphi}$ the cone of the induced $*$ -homomorphism $id \rtimes \varphi : \hat{\mathbb{G}} \rtimes_{\alpha, r} A \longrightarrow \hat{\mathbb{G}} \rtimes_{\beta, r} B$.

Proof. First, recall the definitions of our cones:

$$C_\varphi := \{(a, h) \in A \times C_0((0, 1], B) \mid \varphi(a) = h(1)\} \text{ and}$$

$$C_{id \rtimes \varphi} := \{(X, \tilde{h}) \in \hat{\mathbb{G}} \rtimes_{\alpha, r} A \times C_0((0, 1], \hat{\mathbb{G}} \rtimes_{\beta, r} B) \mid id \rtimes \varphi(X) = \tilde{h}(1)\}.$$

Observe that if (A, α) , (B, β) are $\hat{\mathbb{G}}$ - C^* -algebras, then (C_φ, δ) is again a $\hat{\mathbb{G}}$ - C^* -algebra in the obvious way.

In order to show the canonical $*$ -isomorphism $\hat{\mathbb{G}} \rtimes_r C_\varphi \cong C_{id \rtimes \varphi}$, we are going to show that the C^* -algebra $C_{id \rtimes \varphi}$ satisfies the universal property of the reduced crossed product $\hat{\mathbb{G}} \rtimes_r C_\varphi$. To do so, we have to define a triple $(\bar{\rho}, \bar{V}, \bar{E})$ associated to $C_{id \rtimes \varphi}$ in the sense of Theorem 2.3.1

Given the reduced crossed products $\hat{\mathbb{G}} \rtimes_{\alpha, r} A$ and $\hat{\mathbb{G}} \rtimes_{\beta, r} B$, consider the corresponding canonical associated triples $(\pi_\alpha, U^\alpha, E_\alpha)$ and $(\pi_\beta, U^\beta, E_\beta)$, respectively; and define the non-degenerate $*$ -homomorphism $\bar{\rho} : C_\varphi \longrightarrow C_{id \rtimes \varphi}$ by $\bar{\rho}(a, h) := (\pi_\alpha(a), \pi_\beta \circ h)$, for

all $(a, h) \in C_\varphi$; the unitary representation $\bar{V} \in M(c_0(\hat{\mathbb{G}}) \otimes C_{id \rtimes \varphi})$ is defined by the non-degenerate $*$ -homomorphism $\phi_{\bar{V}} : C_m(\mathbb{G}) \longrightarrow M(C_{id \rtimes \varphi})$ as $\phi_{\bar{V}}(c) := (\phi_{U^\alpha}(c) \cdot, \phi_{U^\beta}(c) \cdot)$, for all $c \in C_m(\mathbb{G})$ and observe that by construction we have $\bar{V}_{i,j}^x = ((U^\alpha)_{i,j}^x \cdot, (U^\beta)_{i,j}^x \cdot) \in M(C_{id \rtimes \varphi})$, for all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$. The strict completely positive KSGNS-faithful map $\bar{E} : C_{id \rtimes \varphi} \longrightarrow M(C_\varphi) = \mathcal{L}_{C_\varphi}(C_\varphi)$ is defined by $\bar{E}(X, \tilde{h}) := (E_\alpha(X) \cdot, E_\beta \circ \tilde{h} \cdot)$, for all $(X, \tilde{h}) \in C_{id \rtimes \varphi}$.

To conclude the proof we have to check the following

- i) $\bar{\rho}(a, h) \bar{V}_{i,j}^x = \sum_{k=1}^{\dim(x)} \bar{V}_{i,k}^x \bar{\rho}(\delta_{k,j}^x(a, h))$, for all $(a, h) \in C_\varphi$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$; which is a routine computation.
- ii) \bar{E} is always a KSGNS-faithful map such that $\bar{E}(\bar{\rho}(a, h) \bar{V}_{i,j}^x) = (a, h) \delta_{x,\epsilon}$, for all $(a, h) \in C_\varphi$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$. The formula is straightforward and concerning the KSGNS-faithfulness we are going to exhibit directly the KSGNS-construction for our $\bar{E} : C_{id \rtimes \varphi} \longrightarrow M(C_\varphi) = \mathcal{L}_{C_\varphi}(C_\varphi)$. In order to do so, recall that $(L^2(\mathbb{G}) \otimes A, id, \Upsilon_\alpha)$ and $(L^2(\mathbb{G}) \otimes A, id, \Upsilon_\beta)$ are the KSGNS constructions for E_α and E_β , respectively. First, we need an appropriated Hilbert C_φ -module. Let's take

$$\mathcal{H} := \{(\xi, \eta) \in L^2(\mathbb{G}) \otimes A \times C_0((0, 1], L^2(\mathbb{G}) \otimes B) \mid \eta(1) = \mathcal{U}_\varphi(\xi \otimes id_B)\},$$

where \mathcal{U}_φ is the canonical isometry between $L^2(\mathbb{G}) \otimes A \otimes_\varphi B$ and $L^2(\mathbb{G}) \otimes B$ of Remark 2.1 above.

Next, consider the adjointable operator $\Upsilon : C_\varphi \longrightarrow \mathcal{H}$ defined by $\Upsilon(a, h) := (\Upsilon_\alpha(a), \Upsilon_\beta \circ h)$, for all $(a, h) \in C_\varphi$ and the representation σ (faithful, thanks to the faithfulness of the KSGNS constructions of E_α and E_β) of $C_{id \rtimes \varphi}$ over \mathcal{H} given by $\sigma(X, \tilde{h}) := (X \cdot, \tilde{h} \cdot)$, for all $(X, \tilde{h}) \in C_{id \rtimes \varphi}$.

In this way, it's easy to check that the triple $(\mathcal{G}, \sigma, \Upsilon)$ with $\mathcal{G} := \overline{\text{span}\{\sigma(C_{id \rtimes \varphi}) \Upsilon(C_\varphi)\}}$ is the KSGNS construction of our \bar{E} and then \bar{E} is KSGNS-faithful (observe by the way that our \bar{E} above is defined exactly through Υ by construction).

■

3. QUANTUM SEMIDIRECT PRODUCT

Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and Γ be a discrete group so that Γ is acting on \mathbb{G} by quantum automorphisms with action α . In this situation, we can construct the *quantum semidirect product of \mathbb{G} by Γ* and it is denoted by

$$\mathbb{F} = \Gamma \rtimes_\alpha \mathbb{G},$$

where $C(\mathbb{F}) = \Gamma \rtimes_{\alpha, m} C_m(\mathbb{G})$. By definition of the crossed product by a discrete group we have a unital faithful $*$ -homomorphism $\pi : C_m(\mathbb{G}) \longrightarrow C(\mathbb{F}) \subset \mathcal{L}_A(l^2(\Gamma) \otimes C_m(\mathbb{G}))$

and a group homomorphism $u : \Gamma \longrightarrow \mathcal{U}(C(\mathbb{F}))$ defined by $u_\gamma := \lambda_\gamma \otimes id_{C_m(\mathbb{G})}$, for all $\gamma \in \Gamma$ such that $C(\mathbb{F}) \equiv \Gamma \ltimes_{\alpha, m} C_m(\mathbb{G}) = C^* < \pi(a)u_\gamma : a \in C_m(\mathbb{G}), \gamma \in \Gamma >$. Thus, the co-multiplication Θ of \mathbb{F} is defined by

$$\Theta \circ \pi = (\pi \otimes \pi) \circ \Delta \text{ and } \Theta(u_\gamma) = u_\gamma \otimes u_\gamma, \text{ for all } \gamma \in \Gamma$$

We have $Irr(\mathbb{F}) = \Gamma \bigoplus Irr(\mathbb{G})$ what means precisely that if $y \in Irr(\mathbb{F})$, then there exist unique $\gamma \in \Gamma$ and $x \in Irr(\mathbb{G})$ such that

$$w^y \equiv w^{(\gamma, x)} = v^\gamma \oplus v^x = [v^\gamma]_{13} [v^x]_{23} \in \mathcal{B}(\mathbb{C} \otimes H_x) \otimes C(\mathbb{F}),$$

where $v^\gamma = 1_{\mathbb{C}} \otimes u_\gamma \in \mathbb{C} \otimes C(\mathbb{F})$ and $v^x = (id \otimes \pi)(w^x) \in \mathcal{B}(H_x) \otimes C(\mathbb{F})$.

As a result, we obtain the following decompositions as well $c_0(\hat{\mathbb{F}}) \cong c_0(\Gamma) \otimes c_0(\hat{\mathbb{G}})$ and $W_{\mathbb{F}} = [W_\Gamma]_{13} [W_{\mathbb{G}}]_{23}$.

3.1 Remark. It's important to observe that $\hat{\mathbb{G}}$ and Γ are quantum subgroups of $\hat{\mathbb{F}}$ with canonical surjections given respectively by $\rho_{\hat{\mathbb{G}}} := \varepsilon_\Gamma \otimes id_{c_0(\hat{\mathbb{G}})}$ and $\rho_\Gamma := id_{c_0(\Gamma)} \otimes \varepsilon_{\hat{\mathbb{G}}}$ where ε_Γ denote de co-unit of Γ and $\varepsilon_{\hat{\mathbb{G}}}$ the co-unit of $\hat{\mathbb{G}}$.

As a result of the previous remark, if (A, δ) is any $\hat{\mathbb{F}}$ - C^* -algebra, then $(A, \delta_{\hat{\mathbb{G}}})$ is a $\hat{\mathbb{G}}$ - C^* -algebra with $\delta_{\hat{\mathbb{G}}} = (\rho_{\hat{\mathbb{G}}} \otimes id_A) \circ \delta$ and (A, δ_Γ) is a Γ - C^* -algebra with $\delta_\Gamma = (\rho_\Gamma \otimes id_A) \circ \delta$.

We use the following notations: the canonical triple (in the sense of Theorem 2.3.1) associated to the reduced crossed product $\Gamma \ltimes_{\delta_\Gamma, r} A \subset \mathcal{L}_A(l^2(\Gamma) \otimes A)$ will be denoted by (σ, ν, E) , the one associated to the reduced crossed product $\hat{\mathbb{F}} \ltimes_{\delta_\Gamma, r} A \subset \mathcal{L}_A(L^2(\mathbb{F}) \otimes A)$ will be denoted by $(\pi_\delta, V, E_\delta)$ and the one associated to the reduced crossed product $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A \subset \mathcal{L}_A(L^2(\mathbb{G}) \otimes A)$ will be denoted by $(\pi_{\delta_{\hat{\mathbb{G}}}}, U, E_{\delta_{\hat{\mathbb{G}}}})$.

3.2 Remark. Using the universal property of $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A$ and the notations above, it's straightforward (just restrict the canonical triple $(\pi_\delta, V, E_\delta)$ to \mathcal{C}) to see that if (A, δ) is a $\hat{\mathbb{F}}$ - C^* -algebra, there exists a canonical $*$ -isomorphism

$$\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A \cong C^* < \pi_\delta(a) V_{i,j}^{(e,x)} : a \in A, x \in Irr(\mathbb{G}), i, j = 1, \dots, \dim(x) > \equiv \mathcal{C}$$

Finally, let (A, δ) be a $\hat{\mathbb{F}}$ - C^* -algebra and consider the reduced crossed product $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A$. Using the canonical identifications $\pi_{\delta_{\hat{\mathbb{G}}}}(a) \cong \pi_\delta(a)$ and $U_{i,j}^x \cong V_{i,j}^{(e,x)}$, for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$ given by the remark above and fixing an element $\gamma \in \Gamma$, we put $\tilde{\pi}^\gamma(a) := \pi_\delta(\delta_\gamma^\Gamma(a))$, for all $a \in A$ and

$$\tilde{U}^\gamma := (id_{c_0(\hat{\mathbb{G}})} \otimes \lambda_\Gamma) \left(\bigoplus_{x \in Irr(\mathbb{G})}^{c_0} (id_{\mathcal{B}(H_x)} \otimes \alpha_\gamma) w^x \right) \otimes id_A.$$

Routine computations show that the formula $\tilde{\pi}^\gamma(a) (\tilde{U}^\gamma)_{i,j}^x = \sum_{k=1}^{\dim(x)} (\tilde{U}^\gamma)_{i,k}^x \tilde{\pi}^\gamma(\delta_{k,j}^{(e,x)}(a))$, holds for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$. So, by the universal property of $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A$, we can conclude the existence of an automorphism

$\partial_\gamma : \hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A \longrightarrow \hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A$, for each $\gamma \in \Gamma$. Moreover, the uniqueness of ∂_γ shows that

$\partial_\gamma \circ \partial_{\gamma'} = \partial_{\gamma\gamma'}$, for all $\gamma, \gamma' \in \Gamma$. In other words, there exists an action $\partial : \Gamma \rightarrow \text{Aut}(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A)$ such that $\partial_\gamma(\pi_\delta(a)U_{i,j}^x) = \pi_\delta(\delta_\gamma^\Gamma(a))\alpha_\gamma(U_{i,j}^x)$, for all $\gamma \in \Gamma$.

Using the alternative description $\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A \cong \mathcal{C}$, the canonical triple associated to the reduced crossed product $\Gamma \rtimes_{\partial, r} \mathcal{C} \subset \mathcal{L}_A(l^2(\Gamma) \otimes \mathcal{C})$ will be denoted by $(\varrho, \vartheta, \mathcal{C})$.

4. TORSION PHENOMENA FOR QUANTUM SEMIDIRECT PRODUCTS

First of all, the description of the irreducible representations of $\mathbb{F} = \Gamma \rtimes_\alpha \mathbb{G}$ allows us to give a natural decomposition of its fusion ring so that we can study the *strong* torsion-freeness of $\hat{\mathbb{F}}$ in terms of the *strong* torsion-freeness of Γ and $\hat{\mathbb{G}}$. Namely, it's straightforward to see that $\text{Fus}(\hat{\mathbb{F}}) = \text{Fus}(\Gamma) \otimes \text{Fus}(\hat{\mathbb{G}})$, so that

4.1 Proposition. *Let $\mathbb{F} = \Gamma \rtimes_\alpha \mathbb{G}$ be the quantum semidirect product of \mathbb{G} by Γ . If Γ and $\hat{\mathbb{G}}$ are strong torsion-free, then $\hat{\mathbb{F}}$ is strong torsion-free.*

Proof. If Γ and $\hat{\mathbb{G}}$ are strong torsion-free, then they are in particular torsion-free. So, by Remark 2.2.1, Γ and $\hat{\mathbb{G}}$ can *not* contain finite discrete quantum subgroups. Hence, $\text{Fus}(\Gamma)$ and $\text{Fus}(\hat{\mathbb{G}})$ can *not* contain finite fusion subrings. Hence Theorem 2.2.2 assures that $\text{Fus}(\Gamma) \otimes \text{Fus}(\hat{\mathbb{G}}) = \text{Fus}(\hat{\mathbb{F}})$ is torsion-free. ■

Although this result implies in particular the torsion-freeness of $\hat{\mathbb{F}}$ in the sense of Meyer-Nest (whenever Γ and $\hat{\mathbb{G}}$ are *strong* torsion-free), it's interesting to obtain the torsion-freeness in the sense of Meyer-Nest *directly* from the torsion-freeness in the sense of Meyer-Nest of Γ and $\hat{\mathbb{G}}$. To do this, we use the theory of spectral subspaces recalled in the preliminaries section.

4.2 Theorem. *Let $\mathbb{F} = \Gamma \rtimes_\alpha \mathbb{G}$ be the quantum semidirect product of \mathbb{G} by Γ . If Γ and $\hat{\mathbb{G}}$ are torsion-free, then $\hat{\mathbb{F}}$ is torsion-free.*

Proof. Let A be a unital finite dimensional C^* -algebra equipped with a (right) torsion action of \mathbb{F} , say $\delta : A \rightarrow A \otimes C(\mathbb{F})$.

Let's define $\Lambda := \{\gamma \in \Gamma \mid \exists x \in \text{Irr}(\mathbb{G}) \text{ such that } K_{(\gamma, x)} \neq 0\}$, where $K_{(\gamma, x)}$ denotes the spectral subspace associated to the representation $(\gamma, x) \equiv y \in \text{Irr}(\mathbb{F})$. We claim that Λ is a finite subgroup of Γ . Indeed, Λ is a subgroup of Γ because given $g, h \in \Lambda$, let $X_g \in K_{(g, x_g)}$ and $X_h \in K_{(h, x_h)}$ be some non-zero elements for the corresponding irreducible representations $x_g, x_h \in \text{Irr}(\mathbb{G})$. Put $y_g := (g, x_g), y_h := (h, x_h) \in \text{Irr}(\mathbb{F})$. By virtue of Lemma 2.1.2, there exist an irreducible representation $z \equiv (\gamma, x) \in \text{Irr}(\mathbb{F})$ and an intertwiner $\Phi \in \text{Mor}(z, y_g \oplus y_h)$ such that $X_g \otimes_\Phi X_h \neq 0$ and recall that $X_g \otimes_\Phi X_h \in K_z$ so that $X_g \otimes_{\tilde{\Phi}} X_h \neq 0$ and $X_g \otimes_{\tilde{\Phi}} X_h \in K_{(\gamma, x)}$, where $\tilde{\Phi} = id_C \otimes \Phi$. This shows that $gh = \gamma \in \Lambda$ as required. Moreover, Λ is finite because A is finite dimensional.

Thanks to the torsion-freeness of Γ , Λ is just the trivial subgroup $\{e\}$. Hence, for every $y \in \text{Irr}(\mathbb{F})$, $K_y \neq 0$ implies $y = (e, x)$ for some $x \in \text{Irr}(\mathbb{G})$. Consequently, the spectral decomposition for $A = \mathcal{A}_{\mathbb{F}}$ becomes $A = \bigoplus_{x \in \text{Irr}(\mathbb{G})} \mathcal{A}_{(e, x)} = \mathcal{A}_{\mathbb{G}}$ and the action δ

takes its values on $A \otimes \pi(C_m(\mathbb{G}))$ so that δ is actually an action of \mathbb{G} on A . Since $\hat{\mathbb{G}}$ is torsion-free by assumption, we achieve the conclusion. ■

5. THE BAUM-CONNES PROPERTY FOR QUANTUM SEMIDIRECT PRODUCTS

Let us recall the notations and definitions of the categorical framework of Meyer-Nest adapted to our situation (see [14] or [10] for a complete presentation of the subject).

Consider the equivariant Kasparov categories associated to $\hat{\mathbb{F}}$ and Γ , say $\mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}}$ and $\mathcal{K}\mathcal{K}^{\Gamma}$, respectively; with canonical suspension functors denoted by Σ . From now on, the word *homomorphism* (resp., *isomorphism*) will mean *homomorphism* (resp., *isomorphism*) in the corresponding Kasparov category; it will be a true homomorphism (resp., isomorphism) between C^* -algebras or any Kasparov triple between C^* -algebras (resp., any KK -equivalence between C^* -algebras). Likewise, consider the usual complementary pair of localizing subcategories in $\mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}}$ and $\mathcal{K}\mathcal{K}^{\Gamma}$, say $(\mathcal{L}_{\hat{\mathbb{F}}}, \mathcal{N}_{\hat{\mathbb{F}}})$ and $(\mathcal{L}_{\Gamma}, \mathcal{N}_{\Gamma})$, respectively. In this way, the canonical triangulated functors associated to these complementary pairs will be denoted by (L, N) and (L', N') , respectively. Next, consider the homological functors defining the *quantum* Baum-Connes assembly maps for $\hat{\mathbb{F}}$ and Γ . Namely,

$$\begin{aligned} F : \mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}} &\longrightarrow \mathcal{A}b & F' : \mathcal{K}\mathcal{K}^{\Gamma} &\longrightarrow \mathcal{A}b \\ (A, \delta) &\longmapsto F(A) := K_*(\hat{\mathbb{F}} \ltimes_{\delta, r} A) & (B, \beta) &\longmapsto F'(B) := K_*(\Gamma \ltimes_{\beta, r} B) \end{aligned}$$

The quantum assembly maps for $\hat{\mathbb{F}}$ and for Γ are given by the following natural transformations $\eta^{\hat{\mathbb{F}}} : \mathbb{L}F \longrightarrow F$ and $\eta^{\Gamma} : \mathbb{L}F' \longrightarrow F'$ (remember that $\mathbb{L}F = F \circ L$ and $\mathbb{L}F' = F' \circ L'$).

To formulate the Baum-Connes property for a discrete quantum group, we need $\hat{\mathbb{F}}$ to be *torsion free*. In that case, the subcategory $\mathcal{L}_{\hat{\mathbb{F}}}$ is easily described as the *localizing subcategory of $\mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}}$ generated by the objects of the form $c_0(\hat{\mathbb{F}}) \otimes C$ with C any C^* -algebra in the Kasparov category $\mathcal{K}\mathcal{K}$* . From now on, we assume that the discrete dual $\hat{\mathbb{F}}$ is *torsion free*. Likewise, any *discrete quantum group will be supposed torsion-free* (in particular, for the quantum semidirect product we keep the preceding section in mind).

In this situation, we say that $\hat{\mathbb{F}}$ satisfies the quantum Baum-Connes property (with coefficients) if the natural transformation $\eta^{\hat{\mathbb{F}}} : \mathbb{L}F \longrightarrow F$ is a natural equivalence.

Given any $\hat{\mathbb{F}}$ - C^* -algebra $(A, \delta) \in \text{Obj.}(\mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}})$, we regard it as an object in $\mathcal{K}\mathcal{K}^{\hat{\mathbb{G}}}$ by restricting the action as explained in Remark 3.1, that is, we consider $(A, \delta_{\hat{\mathbb{G}}}) \in \text{Obj.}(\mathcal{K}\mathcal{K}^{\hat{\mathbb{G}}})$. In this way, it's licit to consider the crossed product $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A$. Observe by the way that we have shown in the preceding section that $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A$ is naturally a Γ - C^* -algebra with action ∂ . Consider now $(B, \nu) \in \text{Obj.}(\mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}})$ an other $\hat{\mathbb{F}}$ - C^* -algebra, let $(B, \nu_{\hat{\mathbb{G}}}) \in \text{Obj.}(\mathcal{K}\mathcal{K}^{\hat{\mathbb{G}}})$ be the corresponding restriction and $\hat{\mathbb{G}} \ltimes_{\nu_{\hat{\mathbb{G}}, r}} B$ the corresponding crossed product (which is again a Γ - C^* -algebra). If $\mathcal{X} \in KK^{\hat{\mathbb{F}}}(A, B)$ is a homomorphism in $\mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}}$ between A and B , we regard it as a homomorphism between A and B in $\mathcal{K}\mathcal{K}^{\hat{\mathbb{G}}}$. Then the functoriality of the crossed product assures that there exists a Kasparov triple (via the descendent homomorphism) $\hat{\mathbb{G}} \ltimes_r \mathcal{X} \in KK(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A, \hat{\mathbb{G}} \ltimes_{\nu_{\hat{\mathbb{G}}, r}} B)$ which is a homomorphism between $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A$ and $\hat{\mathbb{G}} \ltimes_{\nu_{\hat{\mathbb{G}}, r}} B$ in $\mathcal{K}\mathcal{K}$. But $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A$ and

$\hat{\mathbb{G}} \ltimes_{\nu_{\hat{\mathbb{G}},r}} B$ are actually Γ - C^* -algebras and we can show that the descent homomorphism yields an *equivariant* Kasparov triple, that is, $\hat{\mathbb{G}} \ltimes_r \mathcal{X} \in KK^\Gamma(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A, \hat{\mathbb{G}} \ltimes_{\nu_{\hat{\mathbb{G}},r}} B)$. For more details about these functorial constructions we send the reader to [19] or [2].

In other words, it's licit to consider the following functor:

$$\begin{aligned} \mathcal{Z} : \mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}} &\longrightarrow \mathcal{K}\mathcal{K}^\Gamma \\ (A, \delta) &\longmapsto \mathcal{Z}(A) := \hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A \end{aligned}$$

5.1 Theorem. *Let $\mathbb{F} = \Gamma \ltimes_\alpha \mathbb{G}$ be a quantum semidirect product.*

i) *(Associativity for the quantum semidirect product) If (A, δ) is a $\hat{\mathbb{F}}$ - C^* -algebra, then there exists a canonical $*$ -isomorphism*

$$\hat{\mathbb{F}} \ltimes_{\delta,r} A \cong \Gamma \ltimes_{\partial,r} \left(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A \right)$$

ii) *The functor \mathcal{Z} is triangulated such that $\mathcal{Z}(\mathcal{L}_{\hat{\mathbb{F}}}) \subset \mathcal{L}_\Gamma$ and $\mathcal{Z}(\mathcal{N}_{\hat{\mathbb{F}}}) \subset \mathcal{N}_\Gamma$*

Proof. i) In order to prove the isomorphism $\hat{\mathbb{F}} \ltimes_{\delta,r} A \cong \Gamma \ltimes_{\partial,r} \mathcal{C}$, we are going to apply the universal property of the reduced crossed product $\hat{\mathbb{F}} \ltimes_{\delta,r} A$. Thus we have to define a triple $(\bar{\rho}, \bar{V}, \bar{E})$ associated to $\Gamma \ltimes_{\partial,r} \mathcal{C}$ (in the sense of Theorem 2.3.1). Namely, let's put $\bar{\rho} : A \longrightarrow \Gamma \ltimes_{\partial,r} \mathcal{C}$ as the composition $\pi_\delta \circ \varrho$; $\bar{V} \in M(c_0(\hat{\mathbb{F}}) \otimes \Gamma \ltimes_{\partial,r} \mathcal{C})$ as the unitary V itself and $\bar{E} : \Gamma \ltimes_{\partial,r} \mathcal{C} \longrightarrow A$ as the composition $\mathcal{E} \circ E_{\delta|}$.

Routine computations show that the triple $(\bar{\rho}, \bar{V}, \bar{E})$ constructed in this way satisfies the appropriated universal property.

ii) First of all, using the Proposition 2.3.2 it's straightforward to see the stability of \mathcal{Z} with respect to the canonical suspension functors of the corresponding Kasparov categories.

Let us show now the subcategories \mathcal{L} are preserved. Since the functor \mathcal{Z} is compatible with countable direct sums, it suffices to show that for every C^* -algebra $C \in \mathcal{K}\mathcal{K}$ we have $\mathcal{Z}(c_0(\hat{\mathbb{F}}) \otimes C) \in \mathcal{L}_\Gamma$. Indeed, recall by the previous section that we have $c_0(\hat{\mathbb{F}}) \cong c_0(\Gamma) \otimes c_0(\hat{\mathbb{G}})$ so that we can write the following

$$\begin{aligned} \mathcal{Z}(c_0(\hat{\mathbb{F}}) \otimes C) &= \hat{\mathbb{G}} \ltimes_r (c_0(\hat{\mathbb{F}}) \otimes C) \cong \hat{\mathbb{G}} \ltimes_r (c_0(\Gamma) \otimes c_0(\hat{\mathbb{G}}) \otimes C) \\ &\stackrel{*}{\cong} c_0(\Gamma) \otimes \left(\hat{\mathbb{G}} \ltimes_r (c_0(\hat{\mathbb{G}}) \otimes C) \right) \end{aligned} \tag{5.1}$$

where, in the identification $(*)$ above, we use the Proposition 2.3.2 as follows: notice that $\hat{\mathbb{G}}$ acts trivially on $c_0(\Gamma)$ and that $c_0(\hat{\mathbb{G}}) \otimes C$ is a $\hat{\mathbb{G}}$ - C^* -algebra with the natural action given by the (co-opposite) co-multiplication $\hat{\Delta}^{cop}$ of $\hat{\mathbb{G}}$. Then it's licit to consider the reduced crossed product $\hat{\mathbb{G}} \ltimes_r (c_0(\hat{\mathbb{G}}) \otimes C)$. But, by virtue of Baaj-Skandalis duality, we can do the canonical identification $\hat{\mathbb{G}} \ltimes_r (c_0(\hat{\mathbb{G}}) \otimes C) \cong c_0(\hat{\mathbb{G}}) \otimes C$ in $\mathcal{K}\mathcal{K}^{\hat{\mathbb{G}}}$ with trivial action, so it can be viewed as an object in $\mathcal{K}\mathcal{K}$.

Taking $B := \hat{\mathbb{G}} \ltimes_r (c_0(\hat{\mathbb{G}}) \otimes C) \in \mathcal{KK}$, the expression (5.1) above yields $\mathcal{Z}(c_0(\hat{\mathbb{F}}) \otimes C) \cong c_0(\Gamma) \otimes B$ with $B \in \mathcal{KK}$. In other words, $\mathcal{Z}(c_0(\hat{\mathbb{F}}) \otimes C)$ is a Γ - C^* -algebra in \mathcal{KK}^Γ induced by the trivial subgroup $\{e\} < \Gamma$.

Finally, let's show that the subcategories \mathcal{N} are preserved. Recall the definitions of the subcategories $\mathcal{N}_{\hat{\mathbb{F}}}$ and \mathcal{N}_Γ : $\mathcal{N}_{\hat{\mathbb{F}}} = \{(A, \delta) \in \mathcal{KK}^{\hat{\mathbb{F}}} \mid A \cong 0 \text{ in } \mathcal{KK}\}$ and $\mathcal{N}_\Gamma = \{(B, \beta) \in \mathcal{KK}^\Gamma \mid \text{Res}_\Lambda^\Gamma(B) \cong 0 \text{ in } \mathcal{KK}^\Lambda, \forall \Lambda \in \mathcal{F}_\Gamma\}$, where \mathcal{F}_Γ denotes the family of all finite subgroups of Γ . But, using again the Baaaj-Skandalis duality, we can describe this subcategories alternatively as follows:

$\mathcal{N}_{\hat{\mathbb{F}}} = \{(A, \delta) \in \mathcal{KK}^{\hat{\mathbb{F}}} \mid \hat{\mathbb{F}} \ltimes_{\delta, r} A \cong 0 \text{ in } \mathcal{KK}\}$ and $\mathcal{N}_\Gamma = \{(B, \beta) \in \mathcal{KK}^\Gamma \mid \Gamma \ltimes_{\beta, r} B \cong 0 \text{ in } \mathcal{KK}\}$. Thus if $(A, \delta) \in \mathcal{N}_{\hat{\mathbb{F}}}$, we have $\hat{\mathbb{F}} \ltimes_{\delta, r} A \cong 0$ in \mathcal{KK} and using the associativity of a quantum semidirect product showed in (i) we can write $\Gamma \ltimes_{\partial, r} (\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A) \cong 0$ in \mathcal{KK} . In other words, $\mathcal{Z}(A) \in \mathcal{N}_\Gamma$, as required. ■

5.2 Remark. Consider the following functors:

$$\mathcal{KK}^{\hat{\mathbb{F}}} \xrightarrow{j_{\hat{\mathbb{F}}}} \mathcal{KK} \text{ and } \mathcal{KK}^{\hat{\mathbb{F}}} \xrightarrow{\mathcal{Z}} \mathcal{KK}^\Gamma \xrightarrow{j_\Gamma} \mathcal{KK},$$

where $j_{\hat{\mathbb{F}}}$ is the descent functor with respect to $\hat{\mathbb{F}}$ and j_Γ is the descent functor with respect to Γ .

The theorem above yields that for every $\hat{\mathbb{F}}$ - C^* -algebra $(A, \delta) \in \text{Obj.}(\mathcal{KK}^{\hat{\mathbb{F}}})$ there exists an isomorphism $\eta_A : \hat{\mathbb{F}} \ltimes_{\delta, r} A \xrightarrow{\sim} \Gamma \ltimes_{\partial, r} (\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A)$ in \mathcal{KK} . Actually, we get a natural equivalence between the functors above. For this, we have to show that given two $\hat{\mathbb{F}}$ - C^* -algebra $(A, \delta), (B, \nu) \in \text{Obj.}(\mathcal{KK}^{\hat{\mathbb{F}}})$ and a Kasparov triple $\mathcal{X} \in KK^{\hat{\mathbb{F}}}(A, B)$, the following diagram in \mathcal{KK} is commutative

$$\begin{array}{ccc} \hat{\mathbb{F}} \ltimes_{\delta, r} A & \xrightarrow{\hat{\mathbb{F}} \ltimes_r \mathcal{X}} & \hat{\mathbb{F}} \ltimes_{\nu, r} B \\ \eta_A \downarrow & & \downarrow \eta_B \\ \Gamma \ltimes_{\partial, r} (\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A) & \xrightarrow{\Gamma \ltimes_r (\hat{\mathbb{G}} \ltimes_r \mathcal{X})} & \Gamma \ltimes_{\partial', r} (\hat{\mathbb{G}} \ltimes_{\nu_{\hat{\mathbb{G}}, r}} B) \end{array}$$

which is a routine computation. Hence, we have canonically $F = F' \circ \mathcal{Z}$.

5.3 Lemma. *For every $\hat{\mathbb{F}}$ - C^* -algebra (A, δ) there exists an invertible element*

$$\psi \in KK^\Gamma(\hat{\mathbb{G}} \ltimes_r L(A), L'(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}}, r}} A))$$

Proof. Let's start with the existence of such a homomorphism in \mathcal{KK}^Γ . Given a $\hat{\mathbb{F}}$ - C^* -algebra $(A, \delta) \in \mathcal{KK}^{\hat{\mathbb{F}}}$, consider the corresponding distinguish triangle with respect

to the complementary pair $(\mathcal{L}_{\hat{\mathbb{F}}}, \mathcal{N}_{\hat{\mathbb{F}}})$, say $\Sigma(N(A)) \longrightarrow L(A) \xrightarrow{u} A \longrightarrow N(A)$. Since $(A, \delta_{\hat{\mathbb{G}}})$ is a $\hat{\mathbb{G}}$ - C^* -algebra, consider $\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A$ as an object in \mathcal{KK}^Γ and thus the corresponding distinguish $(\mathcal{L}_\Gamma, \mathcal{N}_\Gamma)$ -triangle, say

$$\Sigma(N'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A)) \longrightarrow L'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A) \xrightarrow{u'} \hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A \longrightarrow N'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A) \quad (5.2)$$

Let's fix the Γ - C^* -algebra $\hat{\mathbb{G}} \rtimes_r L(A) =: T \in \mathcal{KK}^\Gamma$ and take the long exact sequence associated to the above triangle with respect to the object T . Namely,

$$\begin{aligned} \dots \rightarrow KK^\Gamma(T, \Sigma(N'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A))) &\rightarrow KK^\Gamma(T, L'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A)) \xrightarrow{(u')^*} \\ &\rightarrow KK^\Gamma(T, \hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A) \rightarrow KK^\Gamma(T, N'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A)) \rightarrow \dots \end{aligned}$$

Since $L(A) \in \mathcal{L}_{\hat{\mathbb{F}}}$, the previous lemma guarantees that $T = \hat{\mathbb{G}} \rtimes_r L(A) \in \mathcal{L}_\Gamma$. But, by definition of complementary pair, we have $\mathcal{L}_\Gamma \subset \mathcal{N}_\Gamma^\perp$. In particular, we obtain $KK^\Gamma(T, \Sigma(N'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A))) = (0) = KK^\Gamma(T, N'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A))$, so that the above long exact sequence yields the isomorphism $KK^\Gamma(T, L'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A)) \xrightarrow{(u')^*} KK^\Gamma(T, \hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A)$. Hence, just take $\psi := (u')_*^{-1}(\mathcal{Z}(u))$.

Next let's check that the element $\psi \in KK^\Gamma(\hat{\mathbb{G}} \rtimes_r L(A), L'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A))$ is invertible. Given the distinguish triangle $\Sigma(N(A)) \longrightarrow L(A) \xrightarrow{u} A \longrightarrow N(A)$ in $\mathcal{KK}^{\hat{\mathbb{F}}}$, we obtain the following distinguish triangle in \mathcal{KK}^Γ applying our *triangulated* functor \mathcal{Z} : $\hat{\mathbb{G}} \rtimes_r \Sigma(N(A)) \longrightarrow \hat{\mathbb{G}} \rtimes_r L(A) \xrightarrow{\mathcal{Z}(u)} \hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A \longrightarrow \hat{\mathbb{G}} \rtimes_r N(A)$. Observe that this triangle is a distinguish one with respect to $(\mathcal{L}_\Gamma, \mathcal{N}_\Gamma)$ because \mathcal{Z} preserve the subcategories \mathcal{L} and \mathcal{N} . But the triangle (5.2) above is by definition a distinguish triangle in \mathcal{KK}^Γ with respect to the complementary pair $(\mathcal{L}_\Gamma, \mathcal{N}_\Gamma)$. By uniqueness, there exists an isomorphism of distinguish triangles in \mathcal{KK}^Γ which yields the conclusion. \blacksquare

5.4 Corollary. *For every $\hat{\mathbb{F}}$ - C^* -algebra (A, δ) there exists an isomorphism*

$$\Psi : \mathbb{L}F(A) \xrightarrow{\sim} \mathbb{L}F'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A)$$

such that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{L}F(A) & \xrightarrow{\Psi} & \mathbb{L}F'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A) \\ \eta_A^{\hat{\mathbb{F}}} \downarrow & & \downarrow \eta_{\hat{\mathbb{G}} \rtimes_r A}^\Gamma \\ F(A) & \xlongequal{\quad} & F'(\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A) \end{array} \quad (5.3)$$

where $\eta_A^{\hat{\mathbb{F}}}$ is the assembly map for $\hat{\mathbb{F}}$ with coefficients in A and $\eta_{\hat{\mathbb{G}} \rtimes_r A}^\Gamma$ is the assembly map for Γ with coefficients in $\hat{\mathbb{G}} \rtimes_{\delta_{\hat{\mathbb{G}}, r}} A$.

Proof. Just take $\Psi := F'(\psi) : K_*\left(\Gamma \ltimes_r \left(\hat{\mathbb{G}} \ltimes_r L(A)\right)\right) \xrightarrow{\sim} K_*(\Gamma \ltimes_r L'(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A))$. The functoriality of constructions and the definition of the element ψ of Lemma 5.3 above yields straightforwardly the diagram of the statement. ■

We can now conclude our study with the following theorem, generalizing and simplifying the result [4] of J. Chabert as we've discussed in the introduction of this article:

5.5 Theorem. *Let $\mathbb{F} = \Gamma \ltimes_{\alpha} \mathbb{G}$ be a quantum semidirect product such that $\hat{\mathbb{F}}$ is a torsion free discrete quantum group and let (A, δ) be any $\hat{\mathbb{F}}$ - C^* -algebra. Then $\hat{\mathbb{F}}$ satisfies the quantum Baum-Connes property with coefficients in A if and only if Γ satisfies the Baum-Connes property with coefficients in $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A$.*

Proof. Fix a $\hat{\mathbb{F}}$ - C^* -algebra (A, δ) . Assume that $\hat{\mathbb{F}}$ satisfies the quantum Baum-Connes property with coefficients in A which means that $K_*(\hat{\mathbb{F}} \ltimes_r L(A)) \xrightarrow{\eta_A^{\hat{\mathbb{F}}}} K_*(\hat{\mathbb{F}} \ltimes_{\delta,r} A)$. By using the associativity for quantum semidirect products and the preceding corollary, we get $K_*(\Gamma \ltimes_r L'(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A)) \xrightarrow{\eta_A^{\hat{\mathbb{F}}} \circ \Psi^{-1}} K_*(\Gamma \ltimes_{\partial,r} (\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A))$, and so $\mathbb{L}F'(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A) \cong F'(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A)$ through $\eta_{\hat{\mathbb{G}} \ltimes_r A}^{\Gamma}$ thanks to the commutativity of the diagram (5.3). That is, Γ satisfies the Baum-Connes property with coefficients in $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A$.

Conversely, assume that Γ satisfies the Baum-Connes property with coefficients in $\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A$ what means that $K_*(\Gamma \ltimes_r L'(\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A)) \xrightarrow{\eta_{\hat{\mathbb{G}} \ltimes_r A}^{\Gamma}} K_*(\Gamma \ltimes_{\partial,r} (\hat{\mathbb{G}} \ltimes_{\delta_{\hat{\mathbb{G}},r}} A))$. By using the associativity for quantum semidirect products and the preceding corollary, we get $K_*(\hat{\mathbb{F}} \ltimes_r L(A)) \xrightarrow{\eta_{\hat{\mathbb{G}} \ltimes_r A}^{\Gamma} \circ \Psi} K_*(\hat{\mathbb{F}} \ltimes_{\delta,r} A)$, and so $\mathbb{L}F(A) \cong F(A)$ through $\eta_A^{\hat{\mathbb{F}}}$ thanks to the commutativity of the diagram (5.3). That is, $\hat{\mathbb{F}}$ satisfies the quantum Baum-Connes property with coefficients in A . ■

To finish we study an other property of own interest: the K -amenability. Namely, we get the following

5.6 Theorem. *Let $\mathbb{F} = \Gamma \ltimes_{\alpha} \mathbb{G}$ be a quantum semidirect product. If \mathbb{G} is co-amenable, then $\hat{\mathbb{F}}$ is K -amenable if and only if Γ is K -amenable.*

Proof. Given the quantum semidirect product \mathbb{F} , consider the canonical surjection $\kappa : C_m(\mathbb{F}) \rightarrow C_r(\mathbb{F})$ where we recall that $C_m(\mathbb{F}) = \Gamma \ltimes_{\alpha,m} C_m(\mathbb{G})$ and $C_r(\mathbb{F}) = \Gamma \ltimes_{\alpha,r} C_r(\mathbb{G})$. Likewise, given the discrete group Γ , consider the canonical surjection $\kappa_{\Gamma} : C_m^*(\Gamma) \rightarrow C_r^*(\Gamma)$.

Since \mathbb{G} is co-amenable, the canonical surjection $\kappa_{\mathbb{G}} : C_m(\mathbb{G}) \rightarrow C_r(\mathbb{G})$ is an $*$ -isomorphism, so we'll write $C(\mathbb{G}) = C_m(\mathbb{G}) = C_r(\mathbb{G})$.

If we assume that Γ is K -amenable, then by virtue of the K -amenability characterization of J. Cuntz (see Theorem 2.1 in [6]), the surjection $\kappa : \Gamma \ltimes_{\alpha,m} C(\mathbb{G}) \rightarrow \Gamma \ltimes_{\alpha,r} C(\mathbb{G})$ induces a KK -equivalence, hence $\hat{\mathbb{F}}$ is K -amenable by definition.

Conversely, suppose that $\hat{\mathbb{F}}$ is K -amenable what means that the canonical surjection $\kappa : C_m(\mathbb{F}) \rightarrow C_r(\mathbb{F})$ induces a KK -equivalence, that is, the corresponding Kasparov triple $[\kappa] \in KK(C_m(\mathbb{F}), C_r(\mathbb{F}))$ is invertible. We have to show that the corresponding Kasparov triple $[\kappa_\Gamma] \in KK(C_m^*(\Gamma), C_r^*(\Gamma))$ is invertible as well.

Given the invertible element $[\kappa] \in KK(C_m(\mathbb{F}), C_r(\mathbb{F}))$, let $\mathcal{X} \in KK(C_r(\mathbb{F}), C_m(\mathbb{F}))$ be its inverse so that we have $[\kappa] \otimes_{C_r(\mathbb{F})} \mathcal{X} = 1_{C_m(\mathbb{F})}$ and $\mathcal{X} \otimes_{C_m(\mathbb{F})} [\kappa] = 1_{C_r(\mathbb{F})}$.

Consider the co-unit $\varepsilon : C_m(\mathbb{G}) = C(\mathbb{G}) = C_r(\mathbb{G}) \rightarrow \mathbb{C}$. So, we get two unital $*$ -homomorphisms $\varphi_m : C_m(\mathbb{F}) \rightarrow C_m^*(\Gamma)$ and $\varphi_r : C_r(\mathbb{F}) \rightarrow C_r^*(\Gamma)$ such that $\kappa_\Gamma \circ \varphi_m = \varphi_r \circ \kappa$.

Recall that $C_m(\mathbb{F}) \equiv \Gamma \ltimes_{\alpha, m} C_m(\mathbb{G}) = C^* \langle \pi(a)u_\gamma : a \in C_m(\mathbb{G}), \gamma \in \Gamma \rangle$. So, with the help of the α -invariant character above, we can identify $C_m^*(\Gamma)$ with the subalgebra of $C_m(\mathbb{F})$ generated by $\{u_\gamma : \gamma \in \Gamma\}$ by universal property (see Remark 3.6 in [8] for more details). Hence, we consider the canonical injection $\iota_m : C_m^*(\Gamma) \hookrightarrow C_m(\mathbb{F})$, so that we have by construction $\varphi_m \circ \iota_m = id_{C_m^*(\Gamma)}$.

Likewise, recall that $C_r(\mathbb{F}) \equiv \Gamma \ltimes_{\alpha, r} C_r(\mathbb{G}) = C^* \langle \pi(a)u_\gamma : a \in C_r(\mathbb{G}), \gamma \in \Gamma \rangle$ is equipped with a GNS-faithful conditional expectation $E : \Gamma \ltimes_{\alpha, r} C_r(\mathbb{G}) \rightarrow C_r(\mathbb{G})$ which restricted to the subalgebra generated by $\{u_\gamma : \gamma \in \Gamma\}$ is just $E(u_\gamma) = \delta_{\gamma, e} \in \mathbb{C}$. Remember as well that $u_\gamma = \lambda_\gamma \otimes id_{C_r(\mathbb{G})} \cong [\lambda_\gamma]_1$ in $\Gamma \ltimes_{\alpha, r} C_r(\mathbb{G}) \subset \mathcal{L}_{C_r(\mathbb{G})}(l^2(\Gamma) \otimes C_r(\mathbb{G}))$; so that this subalgebra will be identified canonically to $C_r^*(\Gamma) = \Gamma \ltimes_{tr, r} \mathbb{C}$ by universal property (here tr denotes the trivial action). Hence, we consider the canonical injection $\iota_r : C_r^*(\Gamma) \hookrightarrow C_r(\mathbb{F})$, so that we have by construction $\varphi_r \circ \iota_r = id_{C_r^*(\Gamma)}$.

Observe that by construction we have as well $\kappa \circ \iota_m = \iota_r \circ \kappa_\Gamma$. Put $\mathcal{Y} := [\iota_r] \otimes_{C_r(\mathbb{F})} \mathcal{X} \otimes_{C_m(\mathbb{F})} [\varphi_m] \in KK(C_r^*(\Gamma), C_m^*(\Gamma))$. Routine computations using all compositions above yields that the element \mathcal{Y} is the inverse of our element $[\kappa_\Gamma]$. ■

6. THE COMPACT BICROSSED PRODUCT CASE

In this section we are going to observe that all preceding results obtained for a quantum semidirect product can be established as well for a compact bicrossed product in the sense of [8]. Let G be a compact group and Γ be a discrete group so that Γ is acting over G on the left with action α and G is acting over Γ on the right with action β such that (Γ, G) is a matched pair. In this situation, we can construct the *compact bicrossed product of the matched pair* (Γ, G) and it is denoted by

$$\mathbb{F} = \Gamma_\alpha \bowtie_\beta G,$$

where $C(\mathbb{F}) = \Gamma \ltimes_{\alpha, m} C(G)$. By definition of the crossed product by a discrete group we have a unital faithful $*$ -homomorphism $\pi : C(G) \rightarrow C(\mathbb{F}) \subset \mathcal{L}_A(l^2(\Gamma) \otimes C(G))$ and a group homomorphism $u : \Gamma \rightarrow \mathcal{U}(C(\mathbb{F}))$ defined by $u_\gamma := \lambda_\gamma \otimes id_{C(G)}$, for all $\gamma \in \Gamma$ such that $C(\mathbb{F}) \equiv \Gamma \ltimes_{\alpha, m} C(G) = C^* \langle \pi(f)u_\gamma : f \in C(G), \gamma \in \Gamma \rangle$. Thus, the

co-multiplication Θ of \mathbb{F} is defined by

$$\Theta \circ \pi = (\pi \otimes \pi) \circ \Delta_G \text{ and } \Theta(u_\gamma) = \sum_{r \in [\gamma]} u_\gamma \alpha(\mathbb{1}_{\gamma,r}) \otimes u_r, \text{ for all } \gamma \in \Gamma,$$

where $\mathbb{1}_{r,s} \equiv \mathbb{1}_{A_{r,s}}$ is the characteristic function of the clopen subset of G defined by $A_{r,s} := \{g \in G : \beta_g(r) = s\}$ for every $r, s \in [\gamma]$. We consider as well the matrix $\left(\mathbb{1}_{r,s}\right)_{r,s \in [\gamma]} \in \mathcal{M}_{\#[\gamma]}(\mathbb{C}) \otimes C(G)$ for each class $[\gamma] \in \Gamma/G$, which is a magic unitary and a unitary representation of G . We send the reader to [8] for more details.

Fima-Mukherjee-Patri's construction allows also to give a concrete description of the representation theory of such a \mathbb{F} . Namely, we have $Irr(\mathbb{F}) = \Gamma/G \bigoplus Irr(G)$, what means precisely that if $y \in Irr(\mathbb{F})$, then there exist unique $[\gamma] \in \Gamma/G$ and $x \in Irr(G)$ such that

$$w^y \equiv w^{([\gamma],x)} = v^{[\gamma]} \oplus v^x = [v^{[\gamma]}]_{13} [v^x]_{23} \in \mathcal{B}(\mathbb{C}^{\#[\gamma]} \otimes H_x) \otimes C(\mathbb{F}),$$

where $v^{[\gamma]} = \sum_{r,s \in [\gamma]} m_{r,s} \otimes u_\gamma \pi(\mathbb{1}_{r,s}) \in \mathcal{M}_{\#[\gamma]}(\mathbb{C}) \otimes C(\mathbb{F})$ and

$v^x = (id \otimes \pi)(w^x) \in \mathcal{B}(H_x) \otimes C(\mathbb{F})$. Observe finally that $w_{p,q}^y = u_\gamma \pi(\mathbb{1}_{r,s}) \pi(w_{i,j}^x)$, for all $p \equiv (r, i)$ and all $q \equiv (s, j)$ with $r, s = 1, \dots, \#[\gamma]$ and $i, j = 1, \dots, \dim(x)$.

6.1 Remark. As in 3.1 we observe that \hat{G} is a quantum subgroup of $\hat{\mathbb{F}}$ with canonical surjection given by $\rho_{\hat{G}} := \varepsilon_\Gamma \otimes id_{C_r^*(G)}$, where ε_Γ is the co-unit of Γ .

As a result of the previous remark, if (A, δ) is any $\hat{\mathbb{F}}\text{-}C^*$ -algebra, then $(A, \delta_{\hat{G}})$ is a $\hat{G}\text{-}C^*$ -algebra with $\delta_{\hat{G}} = (\rho_{\hat{G}} \otimes id_A) \circ \delta$ and replacing the quantum semidirect product $\mathbb{F} = \Gamma \ltimes_\alpha G$ in section 3 by our compact bicrossed product $\mathbb{F} = \Gamma_\alpha \bowtie_\beta G$, we keep on the *same notations* concerning the canonical triples associated to the corresponding reduced crossed products.

6.2 Remark. As in Remark 3.2, it's straightforward to see that if (A, δ) is a $\hat{\mathbb{F}}\text{-}C^*$ -algebra, there exists a canonical $*$ -isomorphism

$$\hat{G} \ltimes_{\delta_{\hat{G}}, r} A \cong C^* \langle \pi_\delta(a) V_{i,j}^{(e,x)} : a \in A, x \in Irr(\mathbb{G}), i, j = 1, \dots, \dim(x) \rangle \equiv \mathcal{C}$$

The main difference with respect to the quantum semidirect product case is the definition of an action of Γ over the reduced crossed product $\hat{G} \ltimes_{\delta_{\hat{G}}, r} A$ such that we get the corresponding associativity decomposition (because in this case $\hat{\Gamma}$ is not longer a quantum subgroup of $\hat{\mathbb{F}}$). This action is established as follows. Use the canonical identifications $\pi_{\delta_{\hat{G}}}(a) \cong \pi_\delta(a)$ and $U_{i,j}^x \cong V_{i,j}^{(e,x)}$, for all $a \in A$, all $x \in Irr(\mathbb{G})$ and all $i, j = 1, \dots, \dim(x)$ given by the remark above and fix an element $\gamma \in \Gamma$. Let's put $\partial_\gamma := Ad(\phi_V(u_\gamma))$, where $Ad(\cdot)$ denotes the adjoint map. This defines clearly an invertible map for each $\gamma \in \Gamma$. Moreover, straightforward computations show that the space $\hat{G} \ltimes_{\delta_{\hat{G}}, r} A$ is preserved. Namely, we get the following formulas

$$\partial_\gamma(\phi_V(w_{i,j}^x)) = \phi_V(\alpha_\gamma(w_{i,j}^x)) \text{ and } \partial_\gamma(\pi_\delta(a)) = \sum_t \phi_V(\mathbb{1}_{r,t}) \pi_\delta \left(\sum_s \delta_{t,s}^{[\gamma] \oplus \epsilon}(a) \right)$$

Moreover, we have $\partial_\gamma \circ \partial_{\gamma'} = \partial_{\gamma\gamma'}$, for all $\gamma, \gamma' \in \Gamma$. In other words, there exists an action $\partial : \Gamma \rightarrow \text{Aut}(\hat{G} \rtimes_{\delta_{\hat{G}}, r} A)$ satisfying the above relations.

Torsion phenomena for a compact bicrossed product

The description of the irreducible representations of such a $\mathbb{F} = \Gamma_\alpha \rtimes_\beta G$ allows us to obtain the decomposition $\text{Fus}(\hat{\mathbb{F}}) = \text{Fus}(\Gamma) \otimes \text{Fus}(G)$ as in the quantum semidirect product case, so that the same argument as in 4.1 yields the following

6.3 Proposition. *Let $\mathbb{F} = \Gamma_\alpha \rtimes_\beta G$ be the compact bicrossed product of the matched pair (Γ, G) . If Γ and G are strong torsion-free, then $\hat{\mathbb{F}}$ is strong torsion-free.*

But we can always do the deeper analysis of the torsion-freeness in the sense of Meyer-Nest as we did in Theorem 4.2. Actually the same arguments hold and we get

6.4 Theorem. *Let $\mathbb{F} = \Gamma_\alpha \rtimes_\beta G$ be the compact bicrossed product of the matched pair (Γ, G) . If Γ and G are torsion-free, then $\hat{\mathbb{F}}$ is torsion-free.*

The Baum-Connes property for a compact bicrossed product

If we analyse the preceding section in which we use systematically the functor \mathcal{Z} of Theorem 5.1 in order to achieve our result, we observe that the key point we employ is the canonical decomposition $\hat{\mathbb{F}} \rtimes_{\delta, r} A \cong \Gamma \rtimes_{\partial, r} (\hat{G} \rtimes_{\delta_{\hat{G}}, r} A)$, for any $\hat{\mathbb{F}}$ - C^* -algebra (A, δ) . But having the action $\partial : \Gamma \rightarrow \text{Aut}(\hat{G} \rtimes_{\delta_{\hat{G}}, r} A)$ constructed at the beginning of this section, we can “copy” the arguments of section 5 replacing the quantum semidirect product $\mathbb{F} = \Gamma \rtimes_\alpha \mathbb{G}$ by a compact bicrossed product $\mathbb{F} = \Gamma_\alpha \rtimes_\beta G$. In particular, the *same* argument of Theorem 5.1 yields

- i) (Associativity for the compact bicrossed product) If (A, δ) is a $\hat{\mathbb{F}}$ - C^* -algebra, then there exists a canonical $*$ -isomorphism

$$\hat{\mathbb{F}} \rtimes_{\delta, r} A \cong \Gamma \rtimes_{\partial, r} (\hat{G} \rtimes_{\delta_{\hat{G}}, r} A)$$

- ii) The functor

$$\begin{aligned} \mathcal{Z} : \mathcal{K} \mathcal{K}^{\hat{\mathbb{F}}} &\longrightarrow \mathcal{K} \mathcal{K}^\Gamma \\ (A, \delta) &\longmapsto \mathcal{Z}(A) := \hat{G} \rtimes_{\delta_{\hat{G}}, r} A \end{aligned}$$

is triangulated such that $\mathcal{Z}(\mathcal{L}_{\hat{\mathbb{F}}}) \subset \mathcal{L}_\Gamma$ and $\mathcal{Z}(\mathcal{N}_{\hat{\mathbb{F}}}) \subset \mathcal{N}_\Gamma$.

6.5 Remark. The same argument used in the preceding section in order to justify the existence of the functor \mathcal{Z} can be applied now and so it's still licit to consider the functor $\mathcal{Z} \mathcal{K} \mathcal{K}^{\hat{\mathbb{F}}} \longrightarrow \mathcal{K} \mathcal{K}^\Gamma$ when $\mathbb{F} = \Gamma_\alpha \rtimes_\beta G$.

Likewise, Remark 5.2 can also be applied; that is, the isomorphism $\hat{\mathbb{F}} \rtimes_{\delta, r} A \cong \Gamma \rtimes_{\partial, r} (\hat{G} \rtimes_{\delta_{\hat{G}}, r} A)$ above yields a natural equivalence between the functors

$\mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}} \xrightarrow{j_{\hat{\mathbb{F}}}} \mathcal{K}\mathcal{K}$ and $\mathcal{K}\mathcal{K}^{\hat{\mathbb{F}}} \xrightarrow{\mathcal{Z}} \mathcal{K}\mathcal{K}^{\Gamma} \xrightarrow{j_{\Gamma}} \mathcal{K}\mathcal{K}$, where $j_{\hat{\mathbb{F}}}$ is the descent functor with respect to $\hat{\mathbb{F}}$ and j_{Γ} is the descent functor with respect to Γ . Hence we have still the identity $F = F' \circ \mathcal{Z}$.

In this way, Theorem 5.5 is also true for such a compact bicrossed product. Namely, we get

6.6 Theorem. *Let $\mathbb{F} = \Gamma_{\alpha} \bowtie_{\beta} G$ be a compact bicrossed product such that $\hat{\mathbb{F}}$ is a torsion free discrete quantum group and let (A, δ) be any $\hat{\mathbb{F}}$ - C^* -algebra. Then $\hat{\mathbb{F}}$ satisfies the quantum Baum-Connes property with coefficients in A if and only if Γ satisfies the Baum-Connes property with coefficients in $\hat{G} \rtimes_{\delta_{\hat{G}}, r} A$.*

The K -amenability property can be as well studied. Namely, the *same* argument of Theorem 5.6 yields the following (notice here that G is automatically co-amenable because it's a classical group):

6.7 Theorem. *Let $\mathbb{F} = \Gamma_{\alpha} \bowtie_{\beta} G$ be a compact bicrossed product. Then $\hat{\mathbb{F}}$ is K -amenable if and only if Γ is K -amenable.*

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